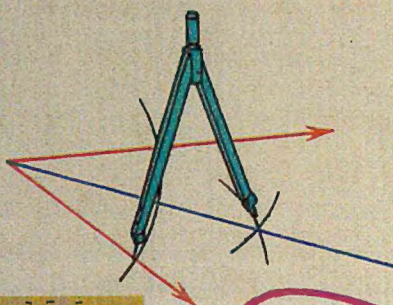
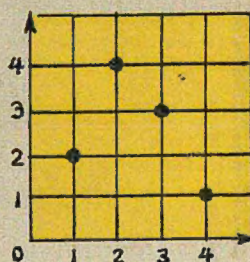
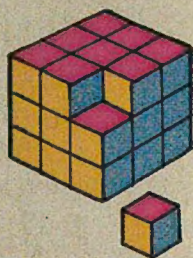


MATHEMATICS

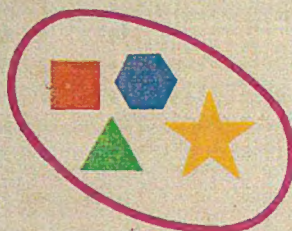
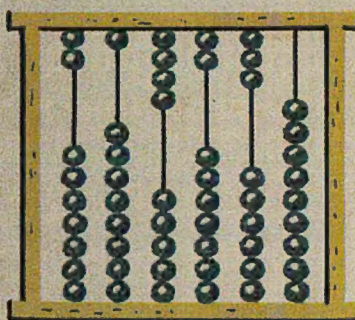
6

3	5	4	
3	5	4	1
6	1	0	8
9	1	5	2
1	2	2	1
1	5	2	5
1	8	3	0
2	1	3	5
2	4	4	0
2	7	4	5



$$\square + 3 = 7$$

$$6 - 1 > 2$$

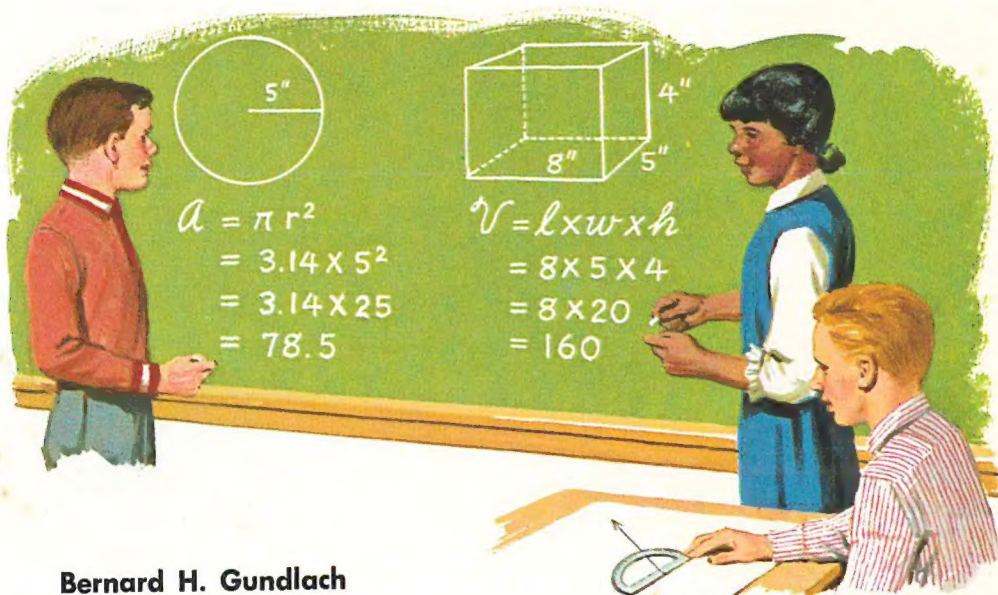


IV

4

MATHEMATICS

6



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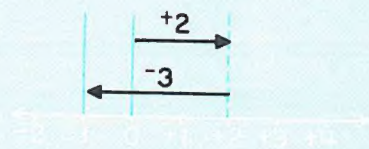
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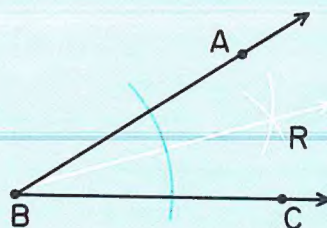
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123
Total



$$+2 + -3 = -1$$



$$m\angle ABR = m\angle RBC$$

$$\angle ABR \cong \angle RBC$$

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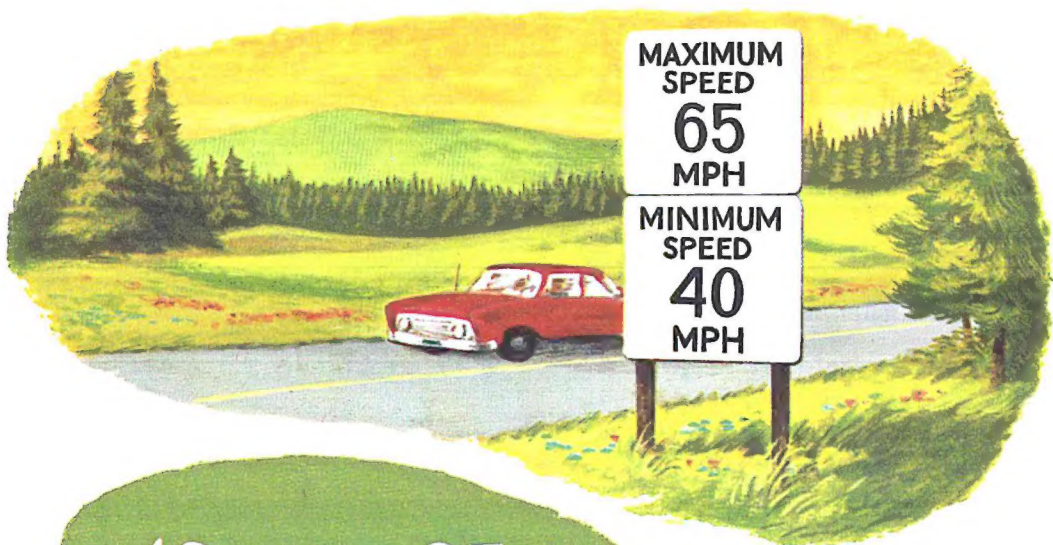
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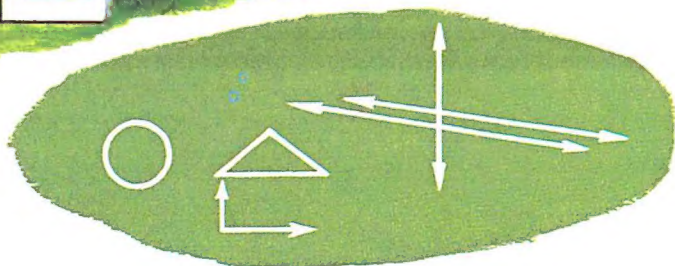
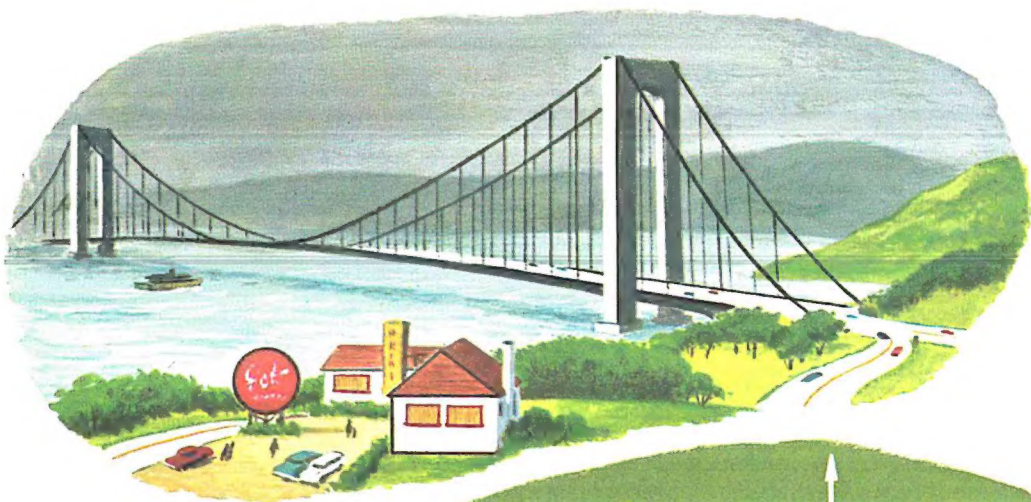
$$V = lwh$$

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$$40 \leq s \leq 65$$



Chapter 1

NUMERATION SYSTEMS

Non-Positional Numeration

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90

The symbols above were used in one of the oldest known numeration systems. About 6000 B.C., the Chinese used these numerals to name numbers.

The numeral $\perp \parallel$ meant $60+2$ or 62. The numeral $\parallel \perp$ meant $2+60$ or 62. The order of the symbols was not important. They added the numbers named by the symbols. Such a system is known as a *non-positional additive* numeration system.

What number is named by $\perp \equiv \Pi$? By $\Pi \perp$?

Oral Use ancient Chinese numerals to answer questions 1–4.

1. What two numerals could be used to name the number 78?
2. Is the order of the symbols in a numeral important?
3. Is there a symbol for the number zero?
4. How is this numeration system similar to the decimal numeration system? How is it different?

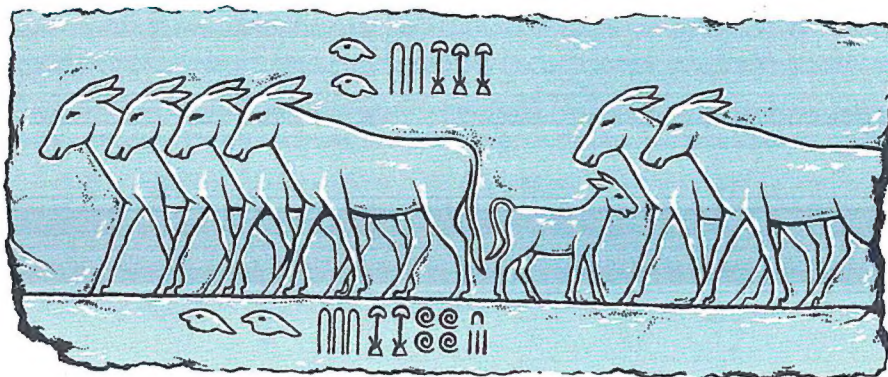
Written Use the ancient Chinese symbols above to write a numeral for each number named below.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
1.	13	25	34	90	68

Write a decimal numeral for each number named below.

	<i>a</i>	<i>b</i>	<i>c</i>
2.	$\equiv \parallel$	$- \parallel \parallel$	$\equiv \Pi \Pi$
3.	$\parallel \perp$	$\top \perp$	$\Pi \equiv$

Egyptian Numeration



The inscription above is from a monument in Cairo. It shows part of the inventory of a king's herds of cattle. The Egyptian numeral above the cattle names the number 223,000. The numeral below the cattle names the number 232,413.

The table below shows the number named by each of the ancient Egyptian symbols.

<i>Egyptian Numeral</i>	<i>Object Pictured</i>	<i>Decimal Numeral</i>
	Tally	1
∩	Heel bone	10
⊙ or ⊗	Coiled rope	100
⋈	Lotus flower	1000
or	Pointed finger	10,000
⋈	Burbot fish	100,000

The Egyptian numeral ∩∩||| meant $10+10+3$ or 23. Since the Egyptian system was a non-positional additive system, twenty-three could also be named by |||∩∩ or |∩∩|| or any other arrangement of two ∩'s and three |'s .

How did they name the number forty-four? Does this system use place value? Does it have a symbol for zero?

Oral Tell the number named by each of the following Egyptian numerals.

- | <i>a</i> | <i>b</i> |
|----------|----------|
| 1. | |
| 2. | |
| 3. | |
| 4. | |
| 5. | |

Tell whether each statement below is *true* or *false*. If a statement is false, tell why it is false.

6. The Egyptian numerals and name the same number.

7. The Egyptians used place value when writing numerals.

8. The numeral means 2 hundreds, no tens, and 2 ones.

9. The numeral means 1 thousand, 2 hundreds, and 3 ones.

10. The Egyptians recognized powers of ten when writing numerals for numbers.

11. The ancient Egyptian numeration system is exactly like the decimal numeration system except that pictures of objects were used instead of 1, 2, 3, and so on.

Written Do the following.

1–5. Write a decimal numeral for each number named in *Oral* 1–5.

Write an Egyptian numeral for each number named below.

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|-----|----------|----------|----------|
| 6. | 27 | 6000 | 12,345 |
| 7. | 48 | 4235 | 20,202 |
| 8. | 90 | 6426 | 31,420 |
| 9. | 421 | 4007 | 231,652 |
| 10. | 374 | 7060 | 102,403 |
| 11. | 706 | 3232 | 314,213 |

Tell how The number *one hundred eleven* can be named by six different Egyptian numerals. Tell how.

Can you do this? The Egyptians also used , an astonished man, to stand for 1 million.

Write an Egyptian numeral for each number named below.

- | | <i>a</i> | <i>b</i> |
|----|-----------|-----------|
| 1. | 1,200,400 | 3,000,521 |
| 2. | 2,570,000 | 4,400,404 |
| 3. | 1,212,720 | 2,345,134 |
| 4. | 3,100,500 | 1,111,111 |

Roman Numeration

You are familiar with Roman numerals like I, II, III, IV, and V. Such numerals are still used on clockfaces, for numbering chapters or volumes of a book, on monuments, and for several other purposes. The Roman numerals have gone through various changes in the course of time. The Roman numerals commonly used today are shown below.

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

Originally, the Roman numeration system was only *additive*. The Romans inherited from the Egyptians the habit of representing numbers in descending order from left to right.

VIII	LXVI	MCCXXXI
$5+3$	$50+10+6$	$1000+200+30+1$
8	66	1231

However, man soon discovered that by using a *subtractive* pattern, he could make the numerals shorter and easier to read. When a symbol for a lesser number preceded a symbol for a greater number, the lesser number was to be subtracted from the greater number.

IV	IX	XL	CM
$5-1$	$10-1$	$50-10$	$1000-100$
4	9	40	900

What number is named by the following Roman numerals?

CCCCCCCCCXXXXXXXXXXIIIIIIII

DCCCCLXXXXVIII

CMXCIX

Which of these numerals is shorter? Which is easier to read? Which of them illustrates the subtractive pattern?

It was agreed that only the symbols for powers of ten (I, X, C, and so on) would be used in the subtractive pattern. Furthermore, only a single symbol would be so used according to the following conditions:

I could be written only before V or X.

X could be written only before L or C.

C could be written only before D or M.

Oral Answer the following questions.

1. Did the Romans have a symbol for zero?

2. How do the Roman numerals show that grouping by tens was used?

3. Is the place-value idea used in Roman numerals?

4. Why is IIX not a permissible numeral for eight?

Tell what number is named by each numeral below.

	<i>a</i>	<i>b</i>	<i>c</i>
5.	VI	XX	XXVI
6.	IV	CX	CXIV
7.	XI	CCC	CCCXI
8.	IX	XL	XLIX
9.	XC	MMD	MMDXC

Written Write a decimal numeral for each number named below.

	<i>a</i>	<i>b</i>
1.	XXXVII	CCLIV
2.	LXXV	MMCDX
3.	CDV	CXXXIX
4.	XLIV	CCCXLIV
5.	CMIX	MCMLXVI

Write a Roman numeral for each number named below.

	<i>a</i>	<i>b</i>	<i>c</i>
6.	12	107	1967
7.	27	423	3429
8.	72	969	2076

Tell how How can you name three different numbers by using each of the symbols L, X, and I once, and only once? What are the numbers?

Computation with Roman Numerals

The Romans, and most of the people of Europe, used the Roman numerals in computation. Sometimes it was necessary to also rely on an abacus or some other calculating device, especially when multiplying or dividing. Let us try using Roman numerals in addition and subtraction.

$$\begin{array}{r} \text{E} \\ \text{CCCXXV} \\ + \text{CCXIII} \\ \hline \text{CCCCCXXXVIII} \\ \text{DXXXVIII} \end{array}$$

$$\begin{array}{r} \text{F} \\ \text{CCCXXV} \\ - \text{CCXIII} \\ \hline \text{CXII} \end{array}$$

The only special feature needed in **E** is that CCCCC is renamed as D. The only special feature needed in **F** is that V is thought of as IIII. How would you state the above examples with decimal numerals?

$$\begin{array}{r} \text{G} \\ \text{DCXLVII} \\ + \text{CCLVIII} \\ \hline \text{CMV} \end{array}$$

$$\begin{array}{r} \text{H} \\ \text{DCXLVII} \\ - \text{CCLVIII} \\ \hline \text{CCCLXXXIX} \end{array}$$

Several combinations must be known to find the result in **G**. What must be known about IIIII? About VV? About LL? About DCCCC? Would it make any difference if you worked from right to left or left to right?

Explain how the result in **H** is obtained.

Translate the examples in **G** and **H** into decimal numerals and check the results. Which is easier for you to work with—Roman numerals or decimal numerals?

Only after multiplication and division became necessary in computation were the Roman numerals gradually abandoned in favor of the decimal system.

Oral Explain how the result is obtained in each example below by using only Roman numerals.

<i>a</i>	<i>b</i>
1. $\begin{array}{r} \text{CCCLVI} \\ + \text{CLXV} \\ \hline \text{DXXI} \end{array}$	$\begin{array}{r} \text{MDCLXIV} \\ - \text{DCCXXIII} \\ \hline \text{CMXLI} \end{array}$
2. $\begin{array}{r} \text{CMLVIII} \\ + \text{CXXIII} \\ \hline \text{MLXXXI} \end{array}$	$\begin{array}{r} \text{CMXXXIV} \\ - \text{DCXXII} \\ \hline \text{CCCXII} \end{array}$

Answer the following questions.

3. In *Oral 1a*, six symbols are used to name the first addend, but only four symbols are needed to name the sum. Can this happen with decimal numerals? Why or why not?

4. In *Oral 1b*, seven symbols are used to name the minuend, but eight symbols are used to name the subtrahend. Can this happen with decimal numerals? Why or why not?

5. Is computation easier for you by using decimal numerals or Roman numerals? What reason can you give for your answer?

Written Do the following.

1–2. Translate *Oral 1–2* into decimal numerals. Perform the addition or the subtraction and check the result of each computation.

Copy. Find each sum or difference. Express each result as a Roman numeral. Then check by using decimal numerals.

<i>a</i>	<i>b</i>
3. $\begin{array}{r} \text{LXIV} \\ + \text{XXII} \\ \hline \end{array}$	$\begin{array}{r} \text{CLXXVI} \\ + \text{CCVIII} \\ \hline \end{array}$
4. $\begin{array}{r} \text{LXIV} \\ - \text{XXII} \\ \hline \end{array}$	$\begin{array}{r} \text{CCLIV} \\ - \text{CLII} \\ \hline \end{array}$
5. $\begin{array}{r} \text{DCLXXVI} \\ + \text{CCLXII} \\ \hline \end{array}$	$\begin{array}{r} \text{CDXXVII} \\ - \text{CCCLIX} \\ \hline \end{array}$
6. $\begin{array}{r} \text{DCLXI} \\ - \text{CCCVII} \\ \hline \end{array}$	$\begin{array}{r} \text{DCCXXXIX} \\ - \text{CCCXXIV} \\ \hline \end{array}$
7. $\begin{array}{r} \text{MCDLVI} \\ + \text{CCLXI} \\ \hline \end{array}$	$\begin{array}{r} \text{MCCCXXVI} \\ + \text{MDCCVI} \\ \hline \end{array}$

Tell why Decide whether each multiplication example below is done correctly. If it is not, tell why not.

1. $\begin{array}{r} \text{XV} \\ \text{V} \\ \hline \text{XXXXX} \end{array}$	$\begin{array}{r} \text{VVVVV} \\ \hline \end{array}$
or $50 + 25 = 75$	
2. $\begin{array}{r} \text{XV} \\ \text{IV} \\ \hline \text{XXXXX} \end{array}$	$\begin{array}{r} \text{VVVVV} \\ \hline \end{array}$
X	V
XXXXXXX VVVVVV or $60 + 30 = 90$	

Can you do this? Find the product of LXXVI and XLIII by using only Roman numerals.

Positional Numeration

Egyptian Numeration

𐪎𐪎𐪎𐪎

𐪎𐪎𐪎𐪎

𐪎𐪎𐪎𐪎

𐪎𐪎𐪎𐪎

𐪎𐪎𐪎𐪎

𐪎𐪎𐪎𐪎

How many different numbers are named by these Egyptian numerals? Is the position of a symbol in an Egyptian numeral important? The Egyptian system is *additive*. What does that mean?

The Egyptian numeration system is called a *non-positional* system.

Decimal Numeration

123

132

213

231

312

321

How many different numbers are named by these decimal numerals? Is the position of a symbol in a decimal numeral important? Is the decimal system additive? Is it additive in the same way as the Egyptian system?

The decimal numeration system is called a *positional* numeration system.

A decimal numeral can also be stated in **expanded notation**. This makes it evident that the decimal system involves both addition and multiplication. Expanded notation also shows the place value of each digit in a numeral.

$$\begin{aligned}
 123 &= (\overset{\text{digits}}{\overbrace{1 \times 100}^{1}}) + (\overset{\text{digits}}{\overbrace{2 \times 10}^{2}}) + (\overset{\text{digits}}{\overbrace{3 \times 1}^{3}}) \\
 &= (1 \times \underset{\text{place values}}{\underbrace{10^2}}) + (2 \times \underbrace{10^1}) + (3 \times \underbrace{10^0})
 \end{aligned}$$

What does 10^2 mean? What does 10^1 mean? What does 10^0 mean?

How would you express 2357 in expanded notation?

Oral Answer the questions below.

1. What number is named by the Chinese numeral $\perp \Pi$? By the Chinese numeral $\Pi \perp$? Is the order of the symbols important?

2. Is the Chinese numeration system positional or non-positional?

3. What number is named by the Roman numeral LX? By the Roman numeral XL? Is the order of the symbols important?

4. Is the Roman numeration system positional or non-positional?

5. Is place value used in either the Chinese or the Roman numeration system?

6. All of the numerals below name the same number. Which of them contains a symbol to denote the number of *tens*?

999IIIIIIII

CCCVIII

308

7. Is it always true that the decimal numeral for a certain number contains fewer symbols than the Egyptian or Roman numeral for that number? Give examples.

8. What advantages are there for using decimal numerals instead of Egyptian or Roman numerals?

Written Do the following.

Change each of the numerals below to expanded notation by naming the place values as powers of ten.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	64	386	2468
2.	97	404	1036
3.	210	4165	75,320
4.	503	3726	86,014

5. Write as many Egyptian numerals as possible by using each of the symbols $\bar{\text{I}}$, $\bar{\text{X}}$, and $\bar{\text{N}}$ once and only once. Write the decimal numeral for each number named.

6. Write as many Roman numerals as possible by using each of the symbols I, V, and X once and only once. Write the decimal numeral for each number named.

7. Write as many decimal numerals as possible by using each of the symbols 2, 3, and 4 once and only once. Use expanded notation to show that each of the numerals names a different number.

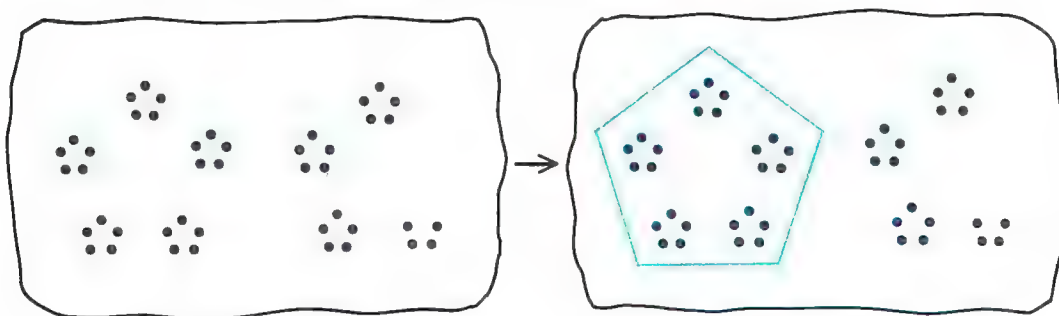
Tell why To name powers of ten as decimal numerals you use the symbol 0, as in 100 and 10,000. Why is a symbol for zero not needed to do this in the Chinese, the Egyptian, and the Roman systems?

Base-Five Numeration

To name the number of members in a set, you could proceed as follows:

- Group the members according to some number.
- Invent symbols to express the number of each group.

Let us invent a place-value numeral for the number of the set shown below. Suppose we group by *fives*.



Form as many subsets of five as possible.

Form as many subsets of five-fives as possible.

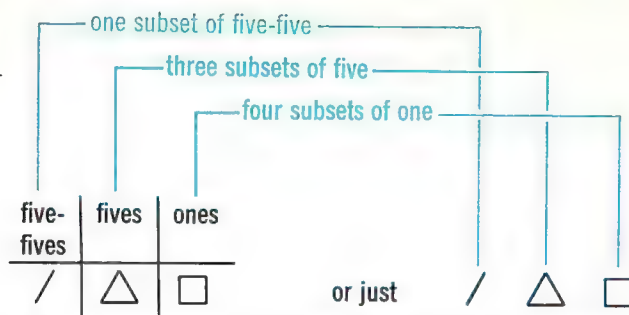
As shown on the right, the set contains *one subset of five-fives*, *three subsets of five*, and *four subsets of one*. After such a grouping, what is the greatest number of single members that could occur? What is the greatest number of subsets of five that could occur? What is the greatest number of subsets of five-fives that could occur?

Let us invent symbols for these numbers as follows.

○	/	∠	△	□
zero	one	two	three	four

How many line segments are there in the symbol ○ ? In the symbol / ? In the symbol ∠ ? Does the pattern continue for △ and □ ? What is an easy way to tell the meaning of each symbol that we invented?

By using the new symbols, you can write a place-value numeral for the number of the set.



Since you grouped the members of the set by *fives*, / Δ □ is called a **base-five numeral**. *Five* is called the **base number** of the numeration system we have invented. It is a **base-five numeration system**.

Oral How would you name the number of each set below by using the new base-five symbols?

- a* *b*
- 1.
 - 2.
 - 3.

Written Write a phrase like “two five-fives, one five, and three ones” for each base-five numeral below. For example, for 1*a* write “three fives and no ones.”

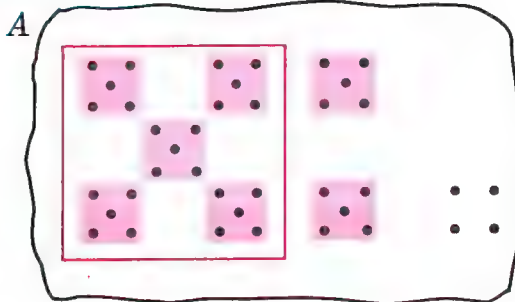
- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|----------|----------|----------|
| 1. | Δ ○ | / ○ Δ | Δ ○ / |
| 2. | ∠ / | ∠ / ∠ | ∠ / □ |
| 3. | □ Δ | □ □ Δ | □ Δ |

Use the new base-five numerals to write a numeral for each of the following.

4. three five-fives, four fives, and two ones
5. two five-fives, no fives, and four ones

Changing from Base Five to Base Ten

The members of set A have been grouped by *fives* and then by *five-fives*. The numeral $/ \angle \square$ names the number of set A .



You can use the symbols 0, 1, 2, 3, and 4 instead of the symbols \bigcirc , $/$, \angle , \triangle and \square to name numbers.

The word *five* at the lower right in 124_{five} indicates that the grouping is by *fives*, or that 124_{five} is a base-five numeral. It is read *one two four, base five*.

five-fives 5×5 5^2	fives 5 5^1	ones 1
$/$	\angle	\square
1	2	4

124_{five}

When no word appears at the lower right of a numeral, it is understood that the grouping is by *tens* and that the numeral is a *base-ten* or a *decimal* numeral.

You can change the base-five numeral 124_{five} to a decimal numeral as shown below.

Base five

124_{five}

Base ten

$$\begin{aligned}
 &= (1 \times 5^2) + (2 \times 5^1) + (4 \times 5^0) \\
 &= (1 \times 25) + (2 \times 5) + (4 \times 1) \\
 &= 25 + 10 + 4 \\
 &= 39
 \end{aligned}$$

Therefore, $124_{\text{five}} = 39$. These two different looking numerals name the same number. You may call 39 *three nine, base ten* or by the shorter wording *thirty-nine*.

We have short number words only in base ten. We have place-value numerals in all bases.

Oral Tell the meaning of each digit in each base-five numeral below.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	23_{five}	231_{five}	333_{five}
2.	41_{five}	304_{five}	1234_{five}
3.	30_{five}	120_{five}	2003_{five}

How would you write a base-five numeral for each number named below in decimal form?

- $(2 \times 5^2) + (3 \times 5^1) + (4 \times 5^0)$
- $(3 \times 5^2) + (0 \times 5^1) + (2 \times 5^0)$
- $(1 \times 5^2) + (2 \times 5^1) + (0 \times 5^0)$
- $(4 \times 5^2) + (0 \times 5^1) + (0 \times 5^0)$

Explain how you would do each of the following.

8. Change 12_{five} to a base-ten numeral.

9. Change 312_{five} to a base-ten numeral.

10. Change 1402_{five} to a base-ten numeral.

Answer the questions below.

11. How many different symbols are needed for base-five numeration?

12. Why is 342 not considered a base-five numeral?

Written Change each base-five numeral below to the simplest decimal numeral for the same number.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	12_{five}	404_{five}	3031_{five}
2.	24_{five}	310_{five}	2142_{five}
3.	32_{five}	221_{five}	1203_{five}
4.	43_{five}	132_{five}	4314_{five}
5.	13_{five}	443_{five}	3410_{five}
6.	20_{five}	314_{five}	2031_{five}
7.	42_{five}	220_{five}	1142_{five}
8.	33_{five}	131_{five}	4203_{five}

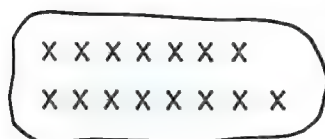
Copy. Replace each \bullet by $<$, $=$, or $>$ so the resulting sentence becomes true.

	<i>a</i>	<i>b</i>
9.	$4_{\text{five}} \bullet 4$	$123_{\text{five}} \bullet 213_{\text{five}}$
10.	$10_{\text{five}} \bullet 10$	$110_{\text{five}} \bullet 101_{\text{five}}$
11.	$21_{\text{five}} \bullet 11$	$100_{\text{five}} \bullet 25$
12.	$31_{\text{five}} \bullet 21$	$100_{\text{five}} \bullet 100$
13.	$44_{\text{five}} \bullet 44$	$204_{\text{five}} \bullet 402_{\text{five}}$

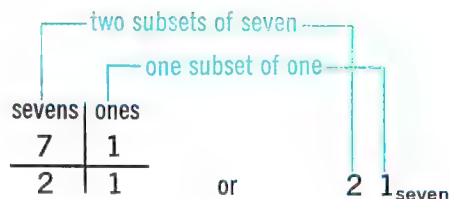
Tell why Why is the numeral 352_{five} not acceptable as a base-five numeral?

Base-Seven Numeration

You can use any whole number greater than one as the base number of a place-value numeration system. Let us name the number of the set below by a base-seven numeral.

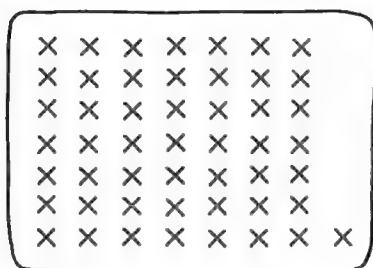


Form as many subsets of seven as possible.

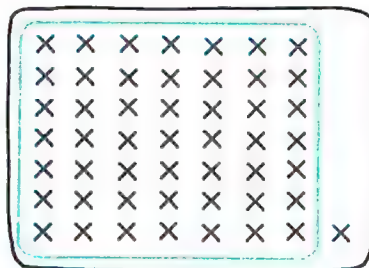


The numeral 21_{seven} is read *two one, base seven*.

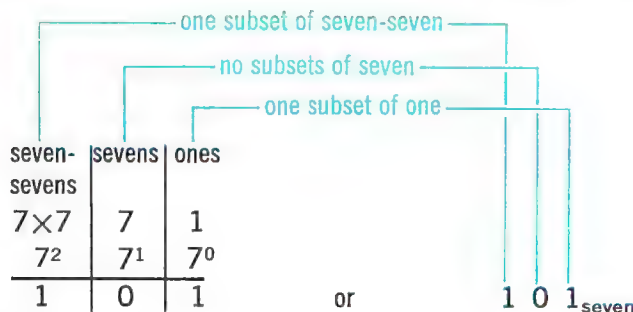
Suppose you wanted to name the number of states in the United States by a base-seven numeral. You might let each x below represent a state and group them as indicated.



Form as many subsets of seven as possible.



Form as many subsets of seven-sevens as possible.



The numeral 101_{seven} is read *one zero one, base seven*.

By using expanded notation you can change any base-seven numeral to a decimal or base-ten numeral.

Base seven

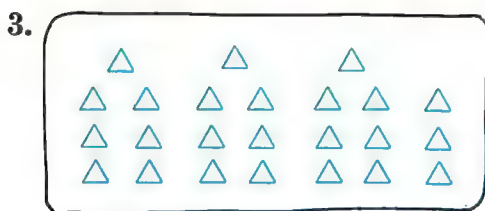
Base ten

101_{seven}

$$\begin{aligned} &= (1 \times 7^2) + (0 \times 7^1) + (1 \times 7^0) \\ &= (1 \times 49) + (0 \times 7) + (1 \times 1) \\ &= 49 + 0 + 1 \\ &= 50 \end{aligned}$$

Therefore, $101_{\text{seven}} = 50$.

Oral What base-seven numeral names the number of each set below?



Read each of the numerals below. Tell what each digit stands for in each numeral.

	<i>a</i>	<i>b</i>	<i>c</i>
4.	10_{seven}	25_{seven}	123_{seven}
5.	23_{seven}	54_{seven}	245_{seven}
6.	40_{seven}	36_{seven}	406_{seven}

Written Do the following.

1. Make a table like that shown below for the whole numbers from one through thirty.

<i>Number</i>	<i>Base Ten</i>	<i>Base Seven</i>
one	1	1
two	2	2
⋮	⋮	⋮
nine	9	12
ten	10	13
⋮	⋮	⋮
twenty-nine	29	41
thirty	30	42

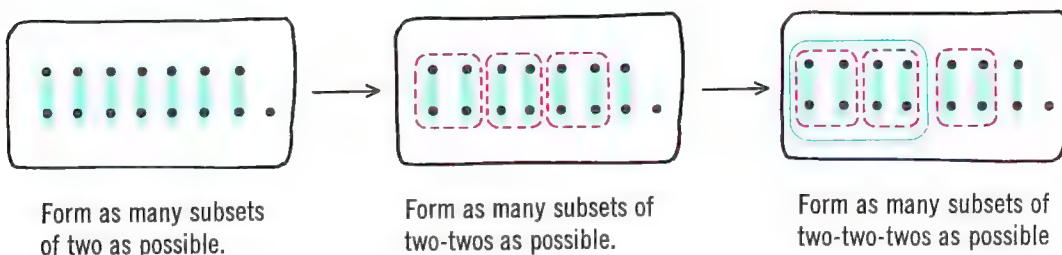
Change each base-seven numeral below to a base-ten numeral.

	<i>a</i>	<i>b</i>	<i>c</i>
2.	32_{seven}	126_{seven}	360_{seven}
3.	54_{seven}	205_{seven}	2153_{seven}

Base-Two Numeration

During the latter part of the seventeenth century a German mathematician, Gottfried Wilhelm von Leibnitz, developed the simplest possible place-value numeration system. He used only the symbols 0 and 1 to develop a *base-two* numeration system.

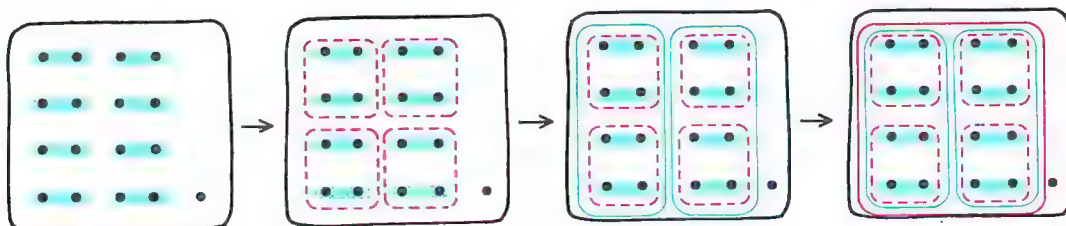
Let us name the number of the set shown below by a base-two numeral.



eights	fours	twos	ones	
$2 \times 2 \times 2$	2×2	2	1	
2^3	2^2	2^1	2^0	
1	1	1	1	or 1 1 1 1 _{two}

The numeral 1111_{two} is read *one one one one, base two*. How would you read 1010_{two} ? What is the difference in meaning between 10 and 10_{two} ?

Explain how subsets are formed below to determine that the number of the set is 10001_{two} .



By using expanded notation you can change any base-two numeral to a decimal or base-ten numeral.

Base two

Base ten

1111_{two}

$$\begin{aligned} &= (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^0) \\ &= (1 \times 8) + (1 \times 4) + (1 \times 2) + (1 \times 1) \\ &= 8 + 4 + 2 + 1 \\ &= 15 \end{aligned}$$

Therefore, $1111_{\text{two}} = 15$.

Oral What base-two numeral names the number of each set below?

1.



2.



3.



Read each of the numerals below. Tell what each digit stands for in each numeral.

a

b

c

4. 10_{two} 101_{two} 1011_{two}

5. 11_{two} 111_{two} 1010_{two}

6. 100_{two} 110_{two} 1110_{two}

Written Do the following.

1. Make a table like that shown below for the whole numbers from one through thirty.

<i>Number</i>	<i>Base Ten</i>	<i>Base Two</i>
one	1	1
two	2	10
⋮	⋮	⋮
seven	7	111
eight	8	1000
⋮	⋮	⋮
twenty-nine	29	11101
thirty	30	11110

Change each base-two numeral below to a base-ten numeral.

a

b

c

2. 110_{two} 1000_{two} 1101_{two}

3. 101_{two} 1011_{two} 1001_{two}

Changing from Base Ten to Base Five

Suppose you had 59 pennies and exchanged them for the fewest number of quarters, nickels, and pennies. How many quarters would you have? How many nickels? How many pennies?

You could then express the number of cents as follows.

$$(2 \times 25) + (1 \times 5) + (4 \times 1)$$

or

$$(2 \times 5^2) + (1 \times 5^1) + (4 \times 5^0) = 214_{\text{five}}$$

Therefore, $59 = 214_{\text{five}}$.

From the above example you can devise a method of changing a base-ten numeral to a base-five numeral.

- Determine the greatest power of five less than or equal to the number.
- Divide the number by that power of five.
- Divide the remainder by the next lesser power of five.
- Repeat step c until the divisor is one.

A convenient arrangement for doing this is shown below.

$5^0 = 1$ $5^1 = 5$ $5^2 = 25$ $5^3 = 125$	<div style="border-left: 1px dashed black; height: 100px; margin-left: 10px;"></div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div style="text-align: right; margin-bottom: 10px;"> $\begin{array}{r} 25 \overline{) 59} \\ \underline{50} \\ 9 \\ \underline{5} \\ 4 \\ \underline{4} \\ 0 \end{array}$ </div> <div style="text-align: right; margin-bottom: 10px;"> $\begin{array}{r} 5 \overline{) 9} \\ \underline{5} \\ 4 \\ \underline{4} \\ 0 \end{array}$ </div> <div style="text-align: right;"> $\begin{array}{r} 1 \overline{) 4} \\ \underline{4} \\ 0 \end{array}$ </div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div style="margin-bottom: 10px;"> $\xrightarrow{\text{2 sets of 25}} 50$ </div> <div style="margin-bottom: 10px;"> $\xrightarrow{\text{1 set of 5}} 5$ </div> <div> $\xrightarrow{\text{4 sets of 1}} +4$ <div style="border-top: 1px solid black; width: 50px; margin: 0 auto; text-align: center;">59</div> </div> </div>
		214_{five}	

Since $25 < 59 < 125$,
the first divisor is 25.

Therefore, $59 = 214_{\text{five}}$.

Notice the order of the quotient digits from top to bottom.
How does this compare with the order of the digits in 214_{five} ?

Explain how 194 is changed to a base-five numeral below.

$$5^0 = 1$$

$$5^1 = 5$$

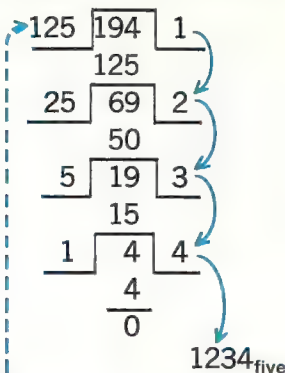
$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

Since $125 < 194 < 625$,
the first divisor is 125.

Therefore, $194 = 1234_{\text{five}}$.



Oral Refer to the example above to answer questions 1–3.

1. Why is it important to use *every* power of 5 less than 5^4 as a divisor?

2. How can you tell when to stop the division process?

3. Since $5^3 < 194$, how can you tell the number of digits in the base-five numeral before doing the division?

Tell the greatest power of five less than each number named below. Then tell how many digits its base-five numeral will have.

	<i>a</i>	<i>b</i>	<i>c</i>
4.	17	121	506
5.	82	136	724

Written Change each base-ten numeral below to a base-five numeral.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	9	64	113	751
2.	11	72	125	647
3.	13	75	156	625
4.	28	94	207	1134
5.	32	99	428	1013
6.	31	93	264	1247

7–12. Check the results of *Written* 1–6 by using expanded notation to change from base five to base ten.

Tell why The ones digit in the numeral 12_{five} names an even number, but 12_{five} names an odd number. Why?

Changing from Base Ten to Other Bases

The method used to change a base-ten numeral to a base-five numeral can be used to change a base-ten numeral to a numeral in any other base. All you need to do is use powers of the new base number instead of powers of five for the divisors.

You can change 141 to a base-seven numeral as shown below.

$7^0 = 1$	$\begin{array}{r} 49 \overline{) 141} 2 \\ \underline{98} \\ 43 6 \\ \underline{42} \\ 1 1 \\ \underline{1} \\ 0 \end{array}$	$2 \text{ sets of } 7^2 = 98$
$7^1 = 7$		$6 \text{ sets of } 7^1 = 42$
$7^2 = 49$		$1 \text{ set of } 7^0 = 1$
$7^3 = 343$		$+1$
	261_{seven}	141

Since $49 < 141 < 343$,
the first divisor is 49.

Therefore, $141 = 261_{\text{seven}}$.

In the above example, why do you use powers of seven? Why is 49 the first divisor? What is the last divisor?

You can change 141 to a base-four numeral as shown below.

$4^0 = 1$	$\begin{array}{r} 64 \overline{) 141} 2 \\ \underline{128} \\ 13 0 \\ \underline{0} \\ 13 3 \\ \underline{12} \\ 1 1 \\ \underline{1} \\ 0 \end{array}$	$2 \text{ sets of } 4^3 = 128$
$4^1 = 4$		$0 \text{ sets of } 4^2 = 0$
$4^2 = 16$		$3 \text{ sets of } 4^1 = 12$
$4^3 = 64$		$1 \text{ set of } 4^0 = 1$
$4^4 = 256$		$+1$
	2031_{four}	141

Since $64 < 141 < 256$,
the first divisor is 64.

Therefore, $141 = 2031_{\text{four}}$.

Why do you use powers of four in this example? Why is 64 the first divisor? What is the last divisor?

Oral Think of changing 163 to a base-seven numeral.

1. First, you would think of the powers of which number?

2. Which of these powers would you use as the first divisor?

3. How many digits will the base-seven numeral have?

Think of changing 29 to a base-two numeral.

4. First, you would think of the powers of which number?

5. Which of these powers would you use as the first divisor?

6. How many digits will the base-two numeral have?

Explain each example below.

7.

$$\begin{array}{r}
 8 \overline{) 14} \overline{) 1} \\
 \underline{8} \\
 4 \overline{) 6} \overline{) 1} \\
 \underline{4} \\
 2 \overline{) 2} \overline{) 1} \\
 \underline{2} \\
 1 \overline{) 0} \overline{) 0} \\
 \underline{0} \\
 0
 \end{array}$$

$$14 = 1110_{\text{two}}$$

b

$$\begin{array}{r}
 27 \overline{) 65} \overline{) 2} \\
 \underline{54} \\
 9 \overline{) 11} \overline{) 1} \\
 \underline{9} \\
 3 \overline{) 2} \overline{) 0} \\
 \underline{0} \\
 1 \overline{) 2} \overline{) 2} \\
 \underline{2} \\
 0
 \end{array}$$

$$65 = 2102_{\text{three}}$$

8. Explain how you would change a base-ten numeral to base six? To base eight?

Written Change each base-ten numeral below to a base-seven numeral.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	12	75	216
2.	52	91	287

Change each base-ten numeral below to a base-two numeral.

	<i>a</i>	<i>b</i>	<i>c</i>
3.	5	11	23
4.	9	19	37

Answer the following questions.

5. John is 53 inches tall. Robert is 53_{seven} inches tall. Who is taller? By how many inches?

6. Alice answered 313_{five} questions correctly out of 100 questions. What was her score in per cent notation?

7. A model car costs 315_{seven} cents. Tom has 303_{five} cents. How many more cents does he need to buy the model car?

8. At the end of the game the score was: Rams 202_{four} and Colts 1000_{three} . Who won the game?

Can you do this? David wrote $122_{\text{■}}$ for the number of states in the United States. The word for the base number became blurred. Can you tell what the word should be?

Practice with Numeration

Part 1 Copy. Write four numerals as indicated for the number of each set.

		<i>Base Two</i>	<i>Base Five</i>	<i>Base Seven</i>	<i>Base Ten</i>
1.	•				
2.	:				
3.	••				
4.	••				
5.	•••				
6.	•••				
7.	••••				
8.	••••				
9.	•••••				
10.	•••••				
11.	•••••				
12.	•••••				
13.	•••••				
14.	•••••				
15.	•••••				
16.	•••••				
17.	•••••				

Part 2 Write an answer for each question below.

1. How many digit symbols are needed for base-five numeration?

2. How many digit symbols are needed for base-two numeration?

3. How many digit symbols are needed for base-ten or decimal numeration?

4. How many digit symbols do you think are needed for base-three numeration? For base-four numeration?

5. If some number b is the base number of a numeration system, how many digit symbols are needed?

Can you do this? To change 132_{five} to a base-seven numeral, you can first change 132_{five} to a base-ten numeral. Then change the resulting base-ten numeral to a base-seven numeral.

Copy and change each numeral as indicated.

- | a | b |
|---|--------------------------------------|
| 1. $132_{\text{five}} = \text{---two}$ | $101_{\text{two}} = \text{---five}$ |
| 2. $23_{\text{seven}} = \text{---five}$ | $24_{\text{five}} = \text{---seven}$ |
| 3. $14_{\text{seven}} = \text{---two}$ | $111_{\text{two}} = \text{---seven}$ |
| 4. $1011_{\text{two}} = \text{---five}$ | $64_{\text{seven}} = \text{---five}$ |

Checkout Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Civilizations in the past developed and used various numeration systems. (5, 6, 8)

2. The decimal numeration system is a positional numeration system. (12)

3. To name the number of a set as a base-five numeral, group the members of the set by fives, five-fives, and so on. (14, 16)

4. Expanded notation can be used to change a numeral in some other base to a base-ten numeral. (16)

5. A base-ten numeral can be changed to some other base by dividing by powers of the base number. (22, 24)

Words to Know

1. Additive numeration system (5, 6)

2. Egyptian numerals (6)

3. Roman numerals (8)

4. Non-positional numeration system, positional numeration system, expanded notation (12)

5. Base-five numeration (14)

6. Base-seven, base-two (18, 20)

Questions to Discuss

1. Why is the decimal system called a positional system? (12)

2. How can you find the decimal numeral for 312_{five} ? (16)

3. How can you find the base-five numeral for 73? (22)

4. How can you find the base-seven numeral for 321? (24)

Written Practice

Change each numeral below to a base-ten numeral.

<i>a</i>	<i>b</i>	<i>c</i>	
1. 21_{five}	143_{five}	200_{five}	(16)

2. 35_{seven}	206_{seven}	1110_{two}	(20)
------------------------	----------------------	---------------------	------

Change each numeral below to a base-five numeral.

<i>a</i>	<i>b</i>	<i>c</i>	
3. 19	47	136	(22)

Change each numeral below to a base-seven numeral.

<i>a</i>	<i>b</i>	<i>c</i>	
4. 55	125	243	(24)

Self-Evaluation

Part 1 Write an Egyptian and a Roman numeral for each number named below.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	23	45	111	1967
2.	97	58	719	3245
3.	29	60	408	5820

Part 2 Find the simplest decimal numeral for each numeral below.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	𐪎𐪎𐪎𐪎	𐪎𐪎𐪎	𐪎𐪎𐪎𐪎
2.	XLIII	MCDL	CCXIX
3.	12 _{five}	46 _{seven}	110 _{two}
4.	430 _{five}	605 _{seven}	1011 _{two}
5.	304 _{five}	316 _{seven}	1101 _{two}
6.	213 _{five}	222 _{seven}	1111 _{two}
7.	3142 _{five}	1025 _{seven}	1000 _{two}
8.	2043 _{five}	2104 _{seven}	11011 _{two}

Part 3 Find each sum or difference.

<i>a</i>	<i>b</i>	<i>c</i>
CXIV	MDCCL	CCLVI
+XII	+CCLV	-CLII

Part 4 Copy and complete the table so that all the numerals in each row name the same number.

	<i>Base Ten</i>	<i>Base Five</i>	<i>Base Seven</i>
1.	—	30 _{five}	—
2.	—	—	24 _{seven}
3.	56	—	—
4.	—	203 _{five}	—
5.	—	—	126 _{seven}
6.	329	—	—
7.	—	1000 _{five}	—
8.	—	—	100 _{seven}

Part 5 Write an answer for each of the following questions.

1. Mary sold 43 tickets and Ed sold 43_{five} tickets. Who sold the greater number of tickets?

2. Pat's score on a test was 75 and Jim's score was 132_{seven}. Who made the better score? By how many points?

3. Which numeral names the greater number: 1101_{two} or 20_{seven}? How much greater is it?

Chapter 2

MATHEMATICAL SENTENCES

Parts of a Mathematical Sentence

Symbols from five sets of symbols may be used to write a mathematical sentence. In advanced mathematics, other symbols are also used.

<i>A</i>	$\{ 0, \frac{1}{2}, 2, 79, 5\frac{1}{4}, .06, 7\%, \dots \}$	Numerals
<i>B</i>	$\{ \square, \triangle, a, b, y, z, n, \dots \}$	Variables or placeholders
<i>C</i>	$\{ =, \neq, <, >, \leq, \geq, \dots \}$	Relation symbols
<i>D</i>	$\{ +, -, \times, \div, \dots \}$	Operation symbols
<i>E</i>	$\{ (), [], \dots \}$	Grouping symbols

Set *A* can be described as the set of all numerals. How would you describe the other sets of symbols?

Oral Tell whether each symbol in each sentence represents a number, a placeholder, a relation, an operation, or a grouping.

a

b

1. $3 + b = 10$

$a + b = b + a$

2. $15 - 7 \neq 12$

$(3 + 2) - n < 8$

3. $k \div 6 > 15$

$17 \times (a \times 7) \leq 1$

Answer the following questions about $3 \times 7 = 21$ and $3 \times (2 + 5) = 21$.

4. How are they alike?

5. How are they different?

Written Write a mathematical sentence by using one or more symbols from the sets specified below.

1. Sets *A* and *C*

2. Sets *A*, *B*, and *C*

3. Sets *A*, *C*, and *D*

4. Sets *A*, *C*, *D*, and *E*

5. All five sets

Tell why From which set must you always choose a symbol when writing a mathematical sentence?

Grouping Symbols

What number is named by the expression below?

$$3 \times 5 + 4$$

One person might first think $3 \times 5 = 15$ and then $15 + 4 = 19$. Another person might first think $5 + 4 = 9$ and then $3 \times 9 = 27$. But this would mean that two numbers are named by the same expression. To avoid such confusion, **grouping symbols** are used to make clear the meaning of any expression. *Parentheses* () and *brackets* [] are two commonly used grouping symbols.

$$(3 \times 5) + 4 \text{ means } 15 + 4 \text{ or } 19$$

$$3 \times (5 + 4) \text{ means } 3 \times 9 \text{ or } 27$$

Sometimes it becomes necessary to use a second pair of grouping symbols inside the first pair. When this occurs, treat the innermost grouping symbols first. To make reading easier, you can use brackets for the outer grouping symbols.

$$\begin{aligned} [(32 \div 4) + 2] \times 6 \\ [8 + 2] \times 6 \\ 10 \times 6 \\ 60 \end{aligned}$$

$$\begin{aligned} (32 \div 4) + (2 \times 6) \\ 8 + 12 \\ 20 \end{aligned}$$

$$\begin{aligned} 32 \div [4 + (2 \times 6)] \\ 32 \div [4 + 12] \\ 32 \div 16 \\ 2 \end{aligned}$$

When part of a mathematical sentence is enclosed by grouping symbols, think of the numeral within the grouping symbols as naming a single number.

If more than one operation symbol is used in a phrase or a sentence, and no grouping symbols are used, let us agree to the following:

- a. If only one operation is indicated, do the computation from left to right.

$$19 - 7 - 5 - 2 \text{ means } [(19 - 7) - 5] - 2$$

- b. If only addition and subtraction are indicated, perform them in order from left to right.

$$7 + 9 - 5 \text{ means } (7 + 9) - 5$$

$$18 - 7 + 4 \text{ means } (18 - 7) + 4$$

- c. If only multiplication and division are indicated, perform them in order from left to right.

$$12 \div 3 \times 2 \text{ means } (12 \div 3) \times 2 \quad 9 \times 4 \div 6 \text{ means } (9 \times 4) \div 6$$

- d. If any other combinations of the operations are indicated, perform the multiplication and division first, and then perform the addition and subtraction.

$$12 \div 4 + 7 \times 3 \text{ means } (12 \div 4) + (7 \times 3)$$

Oral Which operation should you think of first in each of the following expressions? Which operation should you think of second?

<i>a</i>	<i>b</i>
1. $(4+3) \times 5$	$[6+(8 \div 2)] + 5$
2. $4+(3 \times 5)$	$[(6+8) \div 2] + 5$
3. $8 \div (4-2)$	$(6+8) \div (2+5)$
4. $(8 \div 4) - 2$	$6 + [(8 \div 2) + 5]$

Tell how you would decide which number is named by each of the following expressions.

<i>a</i>	<i>b</i>
5. $5 \times 2 + 9 \div 3$	$17 + 4 \times 6 - 20$
6. $8 \times 3 - 16 + 7$	$36 \div 12 \div 3$
7. $12 + 6 \times 5$	$5 \times 3 + 7 \times 1 - 8 \div 2$
8. $17 - 5 + 3$	$29 - 13 - 7$
9. $17 - 5 \times 3$	$4 \times 6 - 12 \div 3$

Written Write the simplest numeral for each number named below.

<i>a</i>	<i>b</i>
1. $2 \times (6+5)$	$[(12 \div 6) + 7] - 1$
2. $(18-8) \div 2$	$(12 \div 6) + (7-1)$
3. $9 + (2 \times 8)$	$(4 \times 3) + (8 \div 2)$
4. $5 + (12 \div 3)$	$[(4 \times 3) + 8] \div 2$
5. $(17-12) + 7$	$4 \times [3 + (8 \div 2)]$

Copy. Insert parentheses or brackets, or both parentheses and brackets, in each expression so that it becomes a name for the number indicated after it.

6. $32 - 12 \div 4 + 2$	Number: 7
7. $32 - 12 \div 4 + 2$	Number: 27
8. $32 - 12 \div 4 + 2$	Number: 30
9. $24 \div 4 + 2 \times 5$	Number: 16
10. $24 \div 4 + 2 \times 5$	Number: 20

Placeholders or Variables

Part of the directions for a game are shown below.

Choose any number from set K . Add five. Then subtract three from the result.

$$K = \{1, 2, 3, 4, 5, 6\}$$

Could you choose 1? Could you choose 2? How many choices do you have?

If you should choose 2, you could write a mathematical expression to show how you followed the directions.

$$(2+5)-3$$

The directions might be stated in a simpler way as a mathematical expression.

$$[(\text{any number from set } K)+5]-3$$

What numerals could replace *any number from set K* ?

To save writing, let us agree to use n to stand for *any number from set K* . Then the directions could be stated as follows.

$$(n+5)-3$$

What numerals can be used to replace n ?

Would it have made any difference if we had agreed to use \square , or a , or t to stand for *any number from set K* ?

When a letter or some other symbol stands for any number from a given set of numbers, the letter or symbol is called a **placeholder** or a **variable**.

The set of numbers whose names may be used as replacements for a variable is called the **replacement set** for that variable.

What is the replacement set for n in $(n+5)-3$?

Oral Answer the questions below.

1. What is a variable?
2. Is a variable to be replaced by a number or by a numeral?
3. If $\{2, 4, 6, 8\}$ is the replacement set for b , how many replacements are there for b ?
4. If the replacement set for n is the set of whole numbers, how many replacements are there for n ?
5. If $\{0, 1, 2, 3, 4\}$ is the replacement set for r in $5+r=7$, how many different sentences are represented by $5+r=7$?
6. In *Oral* 5, if r is replaced by 3, is the resulting sentence true or false?
7. In *Oral* 5, which replacement of r makes the resulting sentence true?

Tell which symbol is the variable in each sentence below.

- | a | b |
|----------------------------|-------------------------|
| 8. $\square \times 4 = 16$ | $(x+2) \times 5 = 31$ |
| 9. $21 \div a = 7$ | $p = (21+4) \times 3$ |
| 10. $d - 12 = 17$ | $c \div 7 = 96 \div 8$ |
| 11. $t + 7 = 42$ | $(15 - e) + 7 = 32$ |
| 12. $81 \div k = 3$ | $7 \times (s - 6) = 56$ |

Written Write a mathematical expression for each set of game directions given below. Use the letter W to denote the set of whole numbers.

1. Choose any number from set W . Multiply that number by 6. Then add 15 to the result.
2. Add any number from set W to 24. Subtract 12 from the result.
3. Divide 32 by any number from set W . Then multiply the result by 8.
4. Subtract 17 from any number in set W . Add 17 to that result.

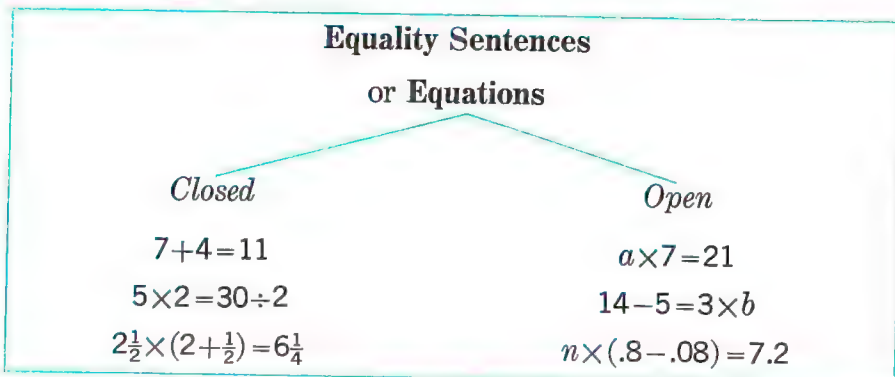
Write a game direction for each mathematical expression below. Let the replacement set be set W .

- | a | b |
|---------------------------|------------------|
| 5. $(n+7) \times 4$ | $(a-5) + 5$ |
| 6. $8 \times (12 \div b)$ | $(4+e) \times 9$ |

Make all the possible replacements for the variable in each expression below. Then find the number named by each resulting expression. Use $\{1, 2, 3\}$ as the replacement set.

- | a | b |
|---------------------|-----------------------|
| 7. $(n+6) \times 3$ | $(7-a) + 8$ |
| 8. $(3-t) + 9$ | $(r \times 6) \div 2$ |
| 9. $(9+k) \times 4$ | $(6 \div t) + 11$ |

Equality Sentences



Does each sentence above contain the symbol $=$? How do you read $=$?

If a mathematical sentence contains the symbol $=$, it is called an **equality sentence** or an **equation**.

If you write two numerals, they might name the same number or they might name different numbers. Do $7+4$ and 11 name the same number? Is $7+4=11$ a true or a false sentence? Do 5×2 and $30 \div 2$ name the same number? Is $5 \times 2 = 30 \div 2$ a true or a false sentence? Is $2\frac{1}{2} \times (2 + \frac{1}{2}) = 6\frac{1}{4}$ a true or a false sentence?

If a mathematical sentence is either true or false, but not both, it is called a **closed sentence**.

Do you know what number is named by $a \times 7$? Can you tell whether $a \times 7 = 21$ is a true or a false sentence? If a were replaced by a numeral, could you tell whether the resulting sentence is true or false? Can you tell whether $14 - 5 = 3 \times b$ is a true or a false sentence? Why or why not? Can you tell whether $n \times (.8 - .08) = 7.2$ is true or false?

If a mathematical sentence is neither true nor false, it is called an **open sentence**.

Does an open sentence have to contain a placeholder? Why or why not?

Oral Answer the following questions.

1. How can you easily tell whether or not a mathematical sentence is an equation?

2. Can an equation be an open sentence? Can it be a closed sentence? Give an example for each answer.

3. Can an equation be a true sentence? Can it be a false sentence? Give an example for each answer.

4. Must an open sentence contain a placeholder? Why or why not?

5. Can a closed sentence contain a placeholder? Why or why not?

Tell whether each sentence below is an open sentence or a closed sentence. If a sentence is closed, tell whether it is true or false.

a	b
6. $3 \times 6 = 15$	$2 \times (5 + 4) = 18$
7. $a + 9 = 17$	$(2 \times 5) + 4 = 18$
8. $81 \div 3 = k$	$6 + (c \times 0) = 12$
9. $24 = 15 + 9$	$7 - (3 \times 2) = n$
10. $56 \div 7 = 9$	$7 = (15 \div 3) + 2$
11. $48 \div 6 = 8$	$8 = 3 + (t - 4)$
12. $m \times 7 = 35$	$(17 - 6) - 5 = 7$

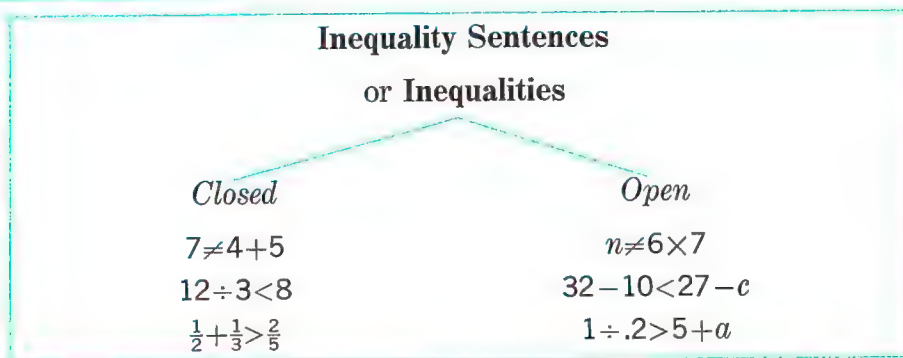
Written Write C for each closed sentence and write O for each open sentence below. Then write *True* or *False* to tell whether each closed sentence is true or false.

a	b
1. $32 \div 4 = 8$	$3 \times (7 + 2) = 27$
2. $64 \div n = 8$	$t + (12 \div 6) = 5$
3. $38 - r = 26$	$w = 21 + 4$
4. $19 + 8 = 23$	$15 - (2 \times 5) = 6$
5. $21 = 7 \times 3$	$13 - (7 - 2) = 4$
6. $b \times 8 = 56$	$k - (7 - 2) = 4$

Copy each open sentence below and replace the letter with a numeral so that the resulting sentence is true. Then copy the sentence again and replace the letter with a numeral so that the resulting sentence is false.

a	b
7. $8 + n = 25$	$n + (3 \times 9) = 41$
8. $72 \div 9 = a$	$7 \times (5 + 4) = b$
9. $r \times 12 = 72$	$(7 \times 5) + 4 = m$
10. $54 - t = 49$	$12 + (9 + 5) = s$
11. $k + 17 = 51$	$(12 + 9) + 5 = z$
12. $49 \div v = 7$	$w - (3 \times 6) = 6$

Inequality Sentences



The sentence $7 \neq 4 + 5$ is read *seven is not equal to four plus five*. Do 7 and $4 + 5$ name the same number? Is $7 \neq 4 + 5$ a true or a false sentence? Is it an open or a closed sentence? Why?

The sentence $12 \div 3 < 8$ is read *twelve divided by three is less than eight*. Does $12 \div 3$ name a lesser number than 8? Is $12 \div 3 < 8$ a true or a false sentence?

The symbol $>$ is read *is greater than*. Does $\frac{1}{2} + \frac{1}{3}$ name a greater number than $\frac{2}{5}$? Is $\frac{1}{2} + \frac{1}{3} > \frac{2}{5}$ a true or a false sentence?

The symbols \neq , $<$, $>$, are some of the **inequality symbols**.

Can you tell whether $n \neq 6 \times 7$ is a true sentence or a false sentence? Why or why not? Is it an open sentence or a closed sentence? Why?

Is $32 - 10 < 27 - c$ an open or a closed sentence? Is $1 \div .2 > 5 + a$ an open or a closed sentence?

If a mathematical sentence contains an inequality symbol, it is called an **inequality sentence** or simply an **inequality**.

In a true inequality sentence, like $3 < 5 \times 2$ or $15 > 8 \div 4$, does the inequality symbol point toward the numeral for the lesser or the greater number?

How can you express $7 > 3$ by using the symbol $<$?

Oral Answer the following questions.

1. How can you easily tell whether or not a mathematical sentence is an inequality sentence?

2. Can an inequality be an open sentence? Can it be a closed sentence? Give an example for each answer.

3. Can an inequality be a true sentence? Can it be a false sentence? Give an example for each answer.

4. Must an open inequality sentence contain a placeholder? Why?

Tell how you would read each of the following inequality sentences. Then tell whether each sentence is an open or a closed sentence.

- | a | b |
|------------------------|--------------------------|
| 5. $8 \times 5 < 19$ | $7 + 5 \neq 6 \times 3$ |
| 6. $a > 12 - 7$ | $7 + 5 < 6 \times 3$ |
| 7. $5 < 4 + 1$ | $7 + 5 > 6 \times 3$ |
| 8. $27 \neq 81 \div 9$ | $4 + t < 17$ |
| 9. $7 > 1 + n$ | $a + b \neq 36$ |
| 10. $k \neq 21 - 13$ | $r \times r > 11$ |
| 11. $15 \div y > 3$ | $7 \times 3 \neq 15 + 6$ |
| 12. $m \neq 24 - 1$ | $12 \div r > 7 - 4$ |

Written Copy. Replace \bullet with either $=$ or \neq so that the resulting sentence is true.

- | a | b |
|---|--|
| 1. $6 \bullet 14 - 8$ | $19 - (2 \times 7) \bullet 10$ |
| 2. $3 \times 6 \bullet 9$ | $2 \times (15 \div 5) \bullet 5$ |
| 3. $2 + 7 \bullet 10$ | $(4 - 1) + 3 \bullet 6$ |
| 4. $\frac{1}{4} + \frac{1}{3} \bullet \frac{1}{12}$ | $\frac{2}{3} + \frac{1}{5} \bullet \frac{2}{15}$ |

Copy. Replace \bullet by $=$, $<$, or $>$ so that the resulting sentence is true.

- | a | b |
|--|--|
| 5. $8 - 3 \bullet 5$ | $17 - 9 \bullet 2 \times 4$ |
| 6. $4 + 8 \bullet 14$ | $(12 - 4) \div 2 \bullet 5$ |
| 7. $9 \bullet 3 \times 2$ | $31 \bullet 3 \times (4 + 5)$ |
| 8. $\frac{1}{2} + \frac{2}{3} \bullet 1$ | $\frac{2}{3} + \frac{1}{5} \bullet \frac{2}{3} \times \frac{1}{5}$ |

Copy. Replace each letter with a numeral for a whole number so that the resulting sentence is true.

- | a | b |
|-----------------------|--------------------|
| 9. $9 + a = 12$ | $18 \div c > 5$ |
| 10. $16 - n < 9$ | $15 \neq d + 7$ |
| 11. $56 \div n > 8$ | $16a = 48$ |
| 12. $m \neq 3(4 + 2)$ | $7 < 36 \div x$ |
| 13. $7 > t \div 4$ | $n \times 14 = 56$ |

Solution Sets of Open Equality Sentences

Open Sentence	Replacement Set	Solution Set
$7 \times n = 28$	$\{0, 2, 4, 6, 8, 10\}$	$\{4\}$

In the example above, which numerals can replace n ? If n is replaced by those numerals, the following closed sentences are obtained.

$$7 \times 0 = 28$$

$$7 \times 4 = 28$$

$$7 \times 8 = 28$$

$$7 \times 2 = 28$$

$$7 \times 6 = 28$$

$$7 \times 10 = 28$$

Which of these closed sentences are true? Which are false? Which replacement or replacements of n make $7 \times n = 28$ become a true sentence?

To *solve an open sentence* means to find all the members of the replacement set that make the given open sentence become a true closed sentence. The set of all such members is called the **solution set** of the open sentence. Each member of the solution set is called a **solution**.

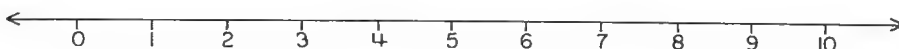
Since 4 is the only member of $\{0, 2, 4, 6, 8, 10\}$ that makes $7 \times n = 28$ become a true sentence, $\{4\}$ is the *solution set* of $7 \times n = 28$. The number 4 is the only *solution*.

Suppose the replacement set is $\{1, 3, 5, 7\}$. Does $7 \times n = 28$ have a solution in $\{1, 3, 5, 7\}$? Since $7 \times n = 28$ becomes a false sentence for every number in the replacement set, the solution set is **the empty set**.

The *empty set* is the set that has no members. It is denoted by either $\{ \}$ or \emptyset .

You can represent a solution set on a number line. Each member of the solution set is indicated by drawing a large dot on the number line.

The solution set $\{4\}$ is shown on the number line below.



Oral Use the following sets as replacement sets to answer the questions 1–5 below.

$$A = \{1, 3, 5, 7, 9\}$$

$$B = \{0, 2, 4, 6, 8\}$$

$$C = \{3, 6, 9, 12, 15\}$$

$$D = \{\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}\}$$

1. Which of the replacement sets above contain a solution for the open sentence $8 - n = 6$? What is that solution?

2. Which of the replacement sets above contain a solution for the open sentence $r \times 7 = 21$? What is that solution?

3. Does set B contain a solution for $m + 5 = 8$? How would you describe the solution set in this case?

4. What is the solution set for $5 + t = 6$ for each replacement set?

5. Is the solution set of an open sentence the same for all replacement sets?

Answer the following questions.

6. If a replacement set does not contain a solution for the open sentence, then the solution set is which set?

7. Is the solution set a subset of the replacement set?

8. How can you show a solution set on a number line?

Written If the number named after each open sentence below is a solution of that open sentence, write *Yes*. If it is not, write *No*.

$$1. \quad 12 + n = 19 \quad 7$$

$$2. \quad 24 \div a = 6 \quad 5$$

$$3. \quad c \times 1.5 = 6.1 \quad 4$$

$$4. \quad b - 9 = 32 \quad 41$$

$$5. \quad (2 \times t) + 12 = 13 \quad \frac{1}{2}$$

$$6. \quad 2 \times (t + 12) = 13 \quad \frac{1}{2}$$

$$7. \quad (a + 13) - 7 = 21 \quad 15$$

$$8. \quad a + (13 - 7) = 21 \quad 15$$

Use $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as the replacement set. Name the solution set for each open sentence below.

a	b
9. $13 + n = 15$	$(3 \times 4) \div 2 = c$

10. $72 \div t = 9$	$(18 \div 6) + b = 9$
---------------------	-----------------------

11. $r \times 5 = 10$	$k + (5 \times 6) = 33$
-----------------------	-------------------------

12. $21 + k = 21$	$g + (7 \times 9) = 81$
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13. $n + 7 = 22$	$(5 + h) \times 4 = 28$
------------------	-------------------------

Tell why The solution set of an open sentence is also called the *Truth Set*. Give a reason why.

Solution Sets of Open Inequality Sentences

Open Sentence	Replacement Set	Solution Set
$3+d < 10$	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$	$\{1, 2, 3, 4, 5, 6\}$

To find the solution set for $3+d < 10$, you could make all the possible replacements for d and check whether the resulting sentences are true or false. However, since $3+7=10$, you have to find a number that *is less than 7* and is also in the replacement set. There are six such numbers in the replacement set. Which numbers are they?

A similar procedure can be used to find the solution set for $2 \times n > 8$. Since $2 \times 4 = 8$, you have to find a number that *is greater than 4* and is also in the replacement set above. What numbers are in the solution set for $2 \times n > 8$?

You can represent the solution set of an open inequality set on a number line. To do this, make large dots on the number line for those points which represent members of the solution set. The solution set $\{1, 2, 3, 4, 5, 6\}$ is shown on the number line below.



The solution set of $2 \times n > 8$ is shown on the number line below.



Why is it unnecessary to extend either of the number lines above beyond the number ten?

Use the replacement above and consider finding the solution set for $2 \times n \neq 8$. Since $2 \times 4 = 8$, which numbers are in the solution set? You can symbolize the solution set by $\{1, 2, 3, 5, 6, 7, 8, 9, 10\}$ or by showing it on a number line as follows.



Oral Answer the following questions.

1. What does it mean to *solve an open sentence*?

2. Can the solution set of an open sentence contain many members? Only one member? No members?

3. State an open sentence and a replacement set so that the solution set will contain:

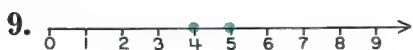
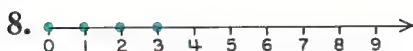
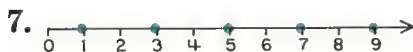
- a. no members
- b. only 1 member
- c. many members

4. If $\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$ is the replacement set, how would you find the solution set for $3\frac{1}{2} + n < 4$?

5. If $\{2, 4, 6, 8\}$ is the replacement set, how would you find the solution set for $9 \times a > 29$?

6. If $\{2, 4, 6, 8\}$ is the replacement set, how would you find the solution set for $17 - r \neq 12$?

Tell what set of numbers is represented by the dots on each number line below.



Written If the number named after each open sentence below is a solution of that open sentence, write *Yes*. If it is not, write *No*.

1. $3 \times t < 26$ 7

2. $k + 18 > 40$ 8

3. $(2 \times n) - 15 < 26$ 9

4. $31 \times (5 + c) > 350$ 5

5. $(a + 16) \times 4 \neq 80$ 4

Use $\{0, 1, 2, 3, 4, 5, 6\}$ as the replacement set. Find the solution set of each open sentence below.

a	b
6. $5 + a < 9$	$(3 \times n) + 1 \neq 7$

7. $4 \times k > 23$	$12 - (5 \times b) = 12$
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8. $4 \times k > 24$	$c < (13 - 9) \div 2$
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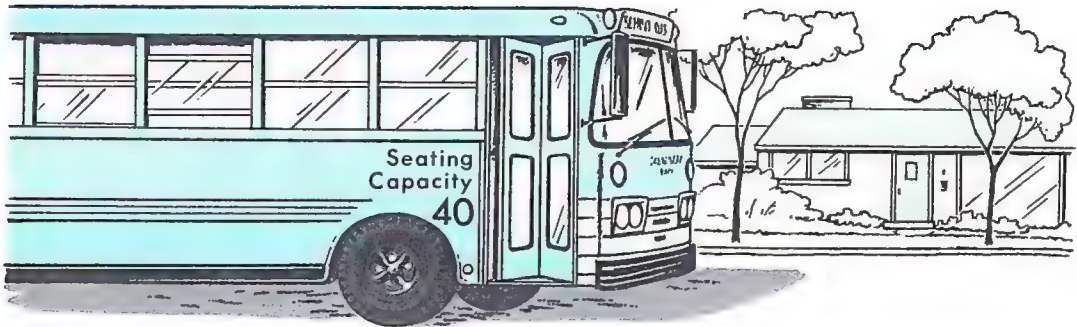
9. $r - 3 \neq 2$	$(36 \div 4) - 1 > h$
-------------------	-----------------------

10–11. Show the solution sets of *Written* 6–7 on number lines.

Can you do this? Part of an open sentence and its solution set are given below. If $\{1, 2, 3, 4\}$ is the replacement set, which of the symbols $<$, $=$, or $>$ should replace \bullet ?

Open sentence	Solution set
1. $6 - n \bullet 2$	$\{4\}$
2. $6 - n \bullet 2$	$\{1, 2, 3\}$

More Inequality Sentences



Can 27 pupils be seated on this bus? Can 40 pupils be seated on this bus? Can 47 pupils be seated on this bus? How can you state how many pupils can be seated on this bus?

Since such situations occur frequently, mathematicians have created symbols to express them. They have combined part of the $=$ sign with either $<$ or $>$ as shown below.

\leq means *is less than or equal to*.

\geq means *is greater than or equal to*.

If r stands for the number of pupils who can be seated in the bus in the above example, then we can write $r \leq 40$.

$r \leq 40$ is an abbreviation for " $r < 40$ or $r = 40$."

$r \leq 40$ becomes a true sentence if r stands for a number that that is either less than 40 or equal to 40.

Why is $27 \leq 40$ a true sentence? Why is $40 \leq 40$ a true sentence? Why is $47 \leq 40$ a false sentence?

$a \geq 25$ is an abbreviation for " $a > 25$ or $a = 25$."

$a \geq 25$ becomes a true sentence if a stands for a number that is either greater than 25 or equal to 25.

Why is $31 \geq 25$ a true sentence? Why is $25 \geq 25$ a true sentence? Why is $19 \geq 25$ a false sentence?

Oral Tell how you would read each of the following sentences.

- | a | b |
|------------------------|-----------------------|
| 1. $a < 7$ | $a \leq 7.4$ |
| 2. $n > 76$ | $n \geq 76$ |
| 3. $k = 13\frac{1}{2}$ | $t \neq \frac{9}{11}$ |
| 4. $26 \leq r$ | $m \geq 0$ |
| 5. $113 \geq s$ | $x \leq 57$ |

Tell whether each sentence below is true or false.

- | a | b |
|------------------|----------------------|
| 6. $13 < 21$ | $3 \times 2 \geq 4$ |
| 7. $13 \leq 21$ | $5 + 4 \leq 9$ |
| 8. $13 \geq 21$ | $8 \times 7 \leq 62$ |
| 9. $13 > 21$ | $75 \div 3 < 25$ |
| 10. $13 \leq 13$ | $75 \div 3 \leq 25$ |

Answer the following questions.

11. What is a short way to state that r stands for a number that is either greater than 6 or equal to 6?

12. If a replacement for b makes $b < 21$ become a true sentence, why will it also make $b \leq 21$ become a true sentence?

Written Write a single open sentence to express the number asked for in each situation below.

1. A compact car is advertised as getting as much as 30 miles per gallon of gasoline. What number of miles per gallon might it get?

2. In order to be on a basketball team, a boy must be at least 66 inches tall. How many inches tall might a boy on the team be?

3. A sign on a boat reads "No more than 4 persons." What number of persons can ride in the boat?

4. You must be at least 21 years of age to vote in Adams County. How many years of age might a voter in Adams County be?

Can you do this? Find the solution set for each open sentence below. Use $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ as the replacement set.

- | a | b |
|---------------|----------------------|
| 1. $a < 5$ | $s + 4 < 7$ |
| 2. $x \leq 5$ | $t + 4 \leq 7$ |
| 3. $r > 6$ | $17 - m > 12$ |
| 4. $c \geq 6$ | $17 - k \geq 12$ |
| 5. $n > 8$ | $56 \div y \leq 8$ |
| 6. $a \leq 1$ | $6 \times c \geq 21$ |

More About Variables

Part of the directions for a game are given below.

Choose any whole number. Multiply the number by 3. Add 10 to that product. Then subtract twice the number chosen.

Let n stand for the number chosen. Then the directions can be expressed as follows.

$$[(3 \times n) + 10] - (2 \times n)$$

In order to shorten this expression, let us agree to represent $3 \times n$ as $3n$ and $2 \times n$ as $2n$.

If no operation symbol occurs between a numeral and a variable, or between two variables, let us agree that the numbers are to be multiplied.

$6k$ means $6 \times k$

ab means $a \times b$

By using the above agreement, you can express the game directions as follows.

$$(3n + 10) - 2n$$

The variable n occurs more than once in this expression. Since n stands for the same number in $3n$ and in $2n$, both n 's must be replaced by names for the same number. For example, if the n in $3n$ is replaced by a name for 5, then the n in $2n$ must also be replaced by a name for 5.

When a replacement for the variable is chosen, you can find the number named by the expression as shown below.

If $n=5$, then

$$\begin{aligned}(3n + 10) - 2n &= [(3 \times 5) + 10] - (2 \times 5) \\ &= (15 + 10) - 10 \\ &= 25 - 10 \\ &= 15.\end{aligned}$$

If $n=2$, then

$$\begin{aligned}(3n + 10) - 2n &= [(3 \times 2) + 10] - (2 \times 2) \\ &= (6 + 10) - 4 \\ &= 16 - 4 \\ &= 12.\end{aligned}$$

Consider the following directions for a game.

Choose two whole numbers. Find their sum.

Let a stand for one of the whole numbers. Let b stand for the other whole number. Their sum can be expressed as follows.

$$a+b$$

Could a and b stand for different numbers? For the same number?

If two or more variables occur in an expression or in an open sentence, they may be replaced by names for the same number or names for different numbers.

Oral Answer the questions below.

1. What operation is understood in the expression $5k$?

2. If the first r in $7r-3r$ is replaced by a name for 4, what must replace the second r ?

3. If a is replaced by 9 in $2a \div b$, must b also be replaced by 9?

Written Find the number named by each expression below if the variable is replaced as indicated.

1. $(5n+6)-3n$ Let $n=7$.

2. $17k+3k$ Let $k=.5$.

3. $4r \times (r-2)$ Let $r=5$.

4. $6m-m$ Let $m=\frac{1}{2}$.

5. $(t+16) \div t$ Let $t=8$.

Find the number named by each expression below if the variables are replaced as follows:

Replace x by 6. Replace y by 2. Replace z by 3.

a	b
6. $x+y$	$x+y+z$

7. $2x \div y$	$xy-z$
----------------	--------

8. $3x+4y$	$x+y-z$
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9. $5x-7y$	$x-y+z$
------------	---------

10. $x+9y$	$2x+y+2z$
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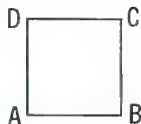
Can you do this? What whole numbers can x and y stand for so that each open sentence below will become a true closed sentence?

a	b
$x+y=8$	$3x+y=12$

If . . . , then . . . Sentences

Set of Rectangles

Set of Squares



Set of prime numbers

$\{2, 3, 5, 7, 11, 13, 17, 19, \dots\}$

- A** If ABCD is a square, then ABCD is a rectangle.
- B** If x names a prime number greater than 2, then x names an odd number.
- C** If $3+4=7$, then $7-4=3$.

You are familiar with sentences like those above. In each of them, two simple sentences are connected by *If . . . , then . . .* to form a new composite sentence. This type of sentence is very important in mathematics.

There are three different things to consider when dealing with an *If . . . , then . . .* sentence.

- Is the *If-part* a true or a false sentence?
- Is the *then-part* a true or a false sentence?
- Is the newly-formed composite sentence true or false?

Later in your study of mathematics you will consider all of the different possible cases. For now, you will deal only with *If . . . , then . . .* sentences in which both simple sentences are true.

In sentence **A** above, is “ABCD is a square” a true or a false sentence? Is “ABCD is a rectangle” a true or a false sentence? Is the composite sentence true or false?

truth of this

implies

the

If ABCD is a square,

then

ABCD is a rectangle.

Do you think that sentence **B** above is true or false? Do you think that sentence **C** above is true or false?

Oral Answer the questions below each *If . . . , then* sentence.

If $6 \times 7 = 42$, then $42 \div 7 = 6$.

1. Is $6 \times 7 = 42$ a true or a false sentence?

2. Is $42 \div 7 = 6$ a true or a false sentence?

3. Is the composite sentence true or false?

If a number is divisible by 2, then it is an even number.

4. Is the *If-part* true for 4? For 12? For 58? For 318?

5. Does any number that makes the *If-part* true also make the *then-part* true?

If $n > 9$, then $n > 5$.

6. Name six numbers that make $n > 9$ become a true sentence. Do all of these numbers also make $n > 5$ become a true sentence?

7. Is the composite sentence true or false?

If $n > 5$, then $n > 9$.

8. Is $n > 5$ true for 6? For 7? For 15? For 267?

9. Do all the numbers named in Oral 8 make $n > 9$ become true?

10. Is the composite sentence true or false?

Written State whether each sentence below is true or false. If your answer is *false*, state a number for which the *If-part* is true and the *then-part* is false.

1. If $r < 4$, then $r < 7$.

2. If $k < 18$, then $k < 13$.

3. If a number is divisible by 3, then it is divisible by 9.

4. If a number is divisible by 9, then it is divisible by 3.

5. If a number names a multiple of five, then its numeral ends with either 0 or 5.

6. If $t \geq 17$, then $t \neq 17$.

7. If a number is a prime number, then it is greater than 1.

Tell how Tell how each of the last 3 sentences below are formed from the first sentence. Then tell whether each sentence is true or false.

1. If ABCD is a square, then ABCD is a rectangle.

2. If ABCD is a rectangle, then ABCD is a square.

3. If ABCD is not a square, then ABCD is not a rectangle.

4. If ABCD is not a rectangle, then ABCD is not a square.

Properties of Equality

You have used the symbol $=$ so often that you may have taken for granted the properties of equality. The properties of equality can be stated and named as shown below.

For all numbers named by a , b , and c :

$$a=a$$

Reflexive property

$$\text{If } a=b, \text{ then } b=a.$$

Symmetric property

$$\text{If } a=b \text{ and } b=c, \text{ then } a=c.$$

Transitive property

The *reflexive property of equality* merely states that every number is equal to itself. For example, $7=7$.

The *symmetric property of equality* allows you to interchange the symbols on either side of $=$. For example, since $2 \times 3 = 6$ you can also write $6 = 2 \times 3$.

The *transitive property of equality* means that two numbers equal to the same number are equal to each other. For example, if $7 \times 8 = x$ and $x = 2 \times 28$, then you can conclude that $7 \times 8 = 2 \times 28$.

Oral Tell whether each of the following sentences is true or false. Then answer the questions below.

$$8 < 8$$

$$\text{If } 3 < 5, \text{ then } 5 < 3.$$

$$\text{If } 2 < 6 \text{ and } 6 < 9, \text{ then } 2 < 9.$$

$$12 > 12$$

$$\text{If } 7 > 4, \text{ then } 4 > 7.$$

$$\text{If } 5 > 3 \text{ and } 3 > 1, \text{ then } 5 > 1.$$

Does inequality have a reflexive property? A symmetric property? A transitive property?

Written Use the symmetric or the transitive property of equality to state a conclusion for each of the following.

1. $5 \times 4 = 20$ and $20 = 10 + 10$, so ____

2. $6(7 + 13) = 120$, so ____

3. If $(a \div 7) + 11 = 17$, then ____

4. If $n^2 = 49$ and $49 = 7 \times 7$, then ____

5. If $\frac{1}{2} \times (7 \times 12) = c$, then ____

6. If $3h = m$ and $m = 21$, then ____

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Grouping symbols are used to make clear the meaning of a mathematical expression. (30)

2. A variable is a letter or some other symbol that stands for any number from a given set. (32)

3. A closed sentence is either true or false. (34)

4. An open sentence is neither true nor false. (34)

5. The symbol \leq means *is less than or equal to*. The symbol \geq means *is greater than or equal to*. (42)

Words to Know

1. Grouping symbols (30)

2. Variable, replacement set (32)

3. Equality sentence, equation, open sentence, closed sentence (34)

4. Inequality symbols, inequality sentence (36)

5. Solution set, solution (38)

6. The empty set (38)

7. If . . . , then . . . sentence (46)

Questions to Discuss

1. In what order should you perform the operations in $8+7-4$? In $5+9\times 3$? (30-31)

2. How can you recognize an equality sentence? An inequality sentence? (34-36)

3. What does it mean to solve an open sentence? (38)

4. How would you find the solution set for $a\times 5>17$ if the replacement set is $\{1,2,3,4,5\}$? (40)

5. If the same variable occurs more than once in an expression, what can you say about the replacements for that variable? (44)

Written Practice

1. Find the number named by $[(12\div 4)\times 7]-5$? (30)

2. Use $\{1,3,5,7,9\}$ as the replacement set to find the solution set for $16+r<22$. (37)

3. Let $a=5$ and $b=7$ to find the number named by $3a+b$? (45)

4. If $5x=y$ and $y=80$, what can you conclude about $5x$ and 80 ? (48)

Self-Evaluation

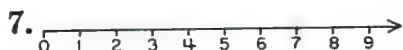
Part 1 Write the simplest numeral for each number named below.

a	b
1. $3 \times (7+5)$	$[(3+2) \times 7] - 5$
2. $(20 \div 4) + 16$	$(3+2) \times (7-5)$
3. $(9 \times 8) - 21$	$3 + [2 \times (7-5)]$
4. $7 + (35 \div 7)$	$3 + [(2 \times 7) - 5]$
5. $(13-9) \div 2$	$[3 + (2 \times 7)] - 5$

Part 2 Use $\{0,1,2,3,4,5,6,7,8\}$ as the replacement set to find the solution set of each open sentence below.

a	b
1. $21 + r = 27$	$(8 \times 6) \div 12 = n$
2. $a \times 15 < 60$	$k > (17-3) \div 2$
3. $28 \div t \neq 7$	$(5+2) - d = 7$
4. $19 - c > 14$	$r \times 1 = r$
5. $b \times 6 \leq 19$	$(21 \div 7) + s > 5$

State the set of numbers shown on each number line below.



Part 3 Write a single open sentence to express the number asked for in each situation.

1. A sign on a storage tank reads "Capacity 520 gallons." What number of gallons might it contain?

2. In order to get a driver's license in a certain state a person must be at least 18 years old. How many years old might a person be who holds a drivers license in that state?

3. A sign on a footbridge reads "Unsafe over 200 pounds." How many pounds might a person weigh and still cross the bridge safely?

Part 4 Find the number named by each expression below if the variables are replaced as follows:

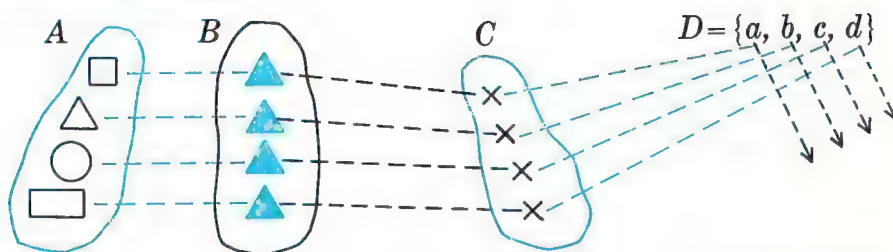
Let $a=3$; let $b=4$; and let $c=5$.

a	b
1. $(4a-9) + a$	$2a + c$
2. $(3 \times 2b) + b$	$6b \div 2a$
3. $7b + (b-2)$	$a + b + c$
4. $(b \times 20) \div b$	$(2a + b) - c$
5. $15c - 4c$	$(4c - 3) \times b$
6. $8a \times (a-2)$	$(7a - 2b) + 3c$

Chapter 3

WHOLE NUMBERS

The Number of a Set



Sets A , B , C , and D are called *equivalent sets*. Describe other sets that are equivalent to each set above. What do all of these sets have in common?

The property that all equivalent sets have in common is called **number**. The number of set B can be abbreviated as $n(B)$.

The number of a set is always a *whole number*.

Set of whole numbers = $\{0, 1, 2, 3, 4, 5, \dots\}$

An important subset of the set of whole numbers is the set of *natural numbers*.

Set of natural numbers = $\{1, 2, 3, 4, 5, \dots\}$

Oral Answer the questions below.

1. How is the set of whole numbers different from the set of natural numbers?

2. What do the three dots in $\{1, 2, 3, 4, 5, \dots\}$ mean?

3. What does the symbol $n(T)$ mean?

Written Write a sentence like $n(A)=4$ for each set below.

1. $R = \{\text{months of the year}\}$

2. $S = \{\text{states in the United States}\}$

3. $T = \{\text{prime numbers less than 20}\}$

4. $Y = \{\text{men over 20 feet tall}\}$

Addition of Whole Numbers

Whole Numbers = $\{0, 1, 2, 3, 4, 5, \dots, 20, 21, 22, 23, 24, \dots\}$

$$2 + 21 = 23$$

Is 2 a member of the set of whole numbers? Is 21 a member of the set of whole numbers? Is the sum of 2 and 21 also a member of the set of whole numbers? Could $2+21$ name some other number, like 221 or $10\frac{1}{2}$, that is different than 23?

Addition is an operation on two numbers of a set resulting in a third number of the set. The numbers being added are called **addends**. The resulting number is called the **sum**.

By studying a few examples of addition, you can discover some of the properties of addition of whole numbers.

$$0+5=5$$

$$17+0=17$$

If zero and any other number are added, what can you say about the sum?

Zero is called the **identity number of addition**. That is, if a represents any whole number, then

$$a+0=0=0+a.$$

Since you already know that $7+5=12$ and $12=5+7$, you can use the transitive property of equality to conclude that $7+5=5+7$.

Does changing the order of the addends change the sum? Since the addends can be commuted, we say that addition of whole numbers is *commutative*.

Commutative Property of Addition: If a and b represent whole numbers, then

$$a+b=b+a.$$

Addition is defined on only two numbers at a time, but it is often necessary to find the sum of more than two numbers. To find the number named by $17+13+42$, you can proceed in either way shown below.

$$\begin{aligned} 17+13+42 &= (17+13)+42 \\ &= 30+42 \\ &= 72 \end{aligned}$$

$$\begin{aligned} 17+13+42 &= 17+(13+42) \\ &= 17+55 \\ &= 72 \end{aligned}$$

Does changing the grouping of the addends change the sum? Since the addends can be associated in either way, we say that addition of whole numbers is *associative*.

Associative Property of Addition: If a , b , and c represent whole numbers, then

$$(a+b)+c=a+(b+c).$$

Oral Tell which property of addition is illustrated by each sentence below.

- $78+39=39+78$
- $1219+0=1219$
- $(17+39)+49=17+(39+49)$
- $(3+61)+18=18+(3+61)$

Tell which numeral should replace the variable so that the resulting sentence is true.

- | a | b |
|------------|-------------------|
| 5. $7+n=7$ | $(1+2)+7=1+(2+r)$ |
| 6. $4+0=k$ | $419+67=c+419$ |
| 7. $t+6=6$ | $s+(3+1)=(2+3)+1$ |

Written Name the property used in each step below to find the simplest numeral for the sum.

- _____ $t=76+(17+24)$
- _____ $= (76+17)+24$
- _____ $= (17+76)+24$
- _____ $= 17+(76+24)$
- _____ $= 17+100$ or 117

Copy. Find the simplest numeral for each sum.

- | a | b |
|---------------|------------|
| 4. $21+27+19$ | $48+67+42$ |
| 5. $54+13+26$ | $25+83+35$ |
| 6. $52+35+48$ | $67+49+33$ |
| 7. $25+89+75$ | $27+59+43$ |
| 8. $18+27+62$ | $86+79+14$ |

The Inverse Operation of Addition

Man has created mathematics with a high degree of consistency and excellence. One aspect of this excellence is that the results of most mathematical operations can be undone, if so desired, by reversing the procedures. In such cases, the original operation and the newly created operation are called *inverse operations*.

To undo an addition result, man has created the operation of subtraction.

$$\text{If } 3+7=10, \text{ then } 3=10-7.$$

$$\text{If } 7+3=10, \text{ then } 7=10-3.$$

How is *adding 7* undone in the first example? How is *adding 3* undone in the second example?

Let a , b , and c represent whole numbers.

$$\text{If } a+b=c, \text{ then } a=c-b \text{ and } b=c-a.$$

Since subtraction is the inverse operation of addition, you should expect that addition is the inverse operation of subtraction.

$$\text{If } 7-x=2, \text{ then } 7=2+x.$$

$$\text{If } y-15=8, \text{ then } y=8+15.$$

What is the inverse of *subtracting x* ? What is the inverse of *subtracting 15*?

Suppose you start with some whole number a and add a whole number b to it. What can you do to the result to obtain the number a with which you started?

Suppose you start with some whole number a and subtract a whole number b from it. What can you do to the result to obtain the number a with which you started?

If a and b represent whole numbers, then

$$(a+b)-b=a \text{ and } (a-b)+b=a.$$

Oral Answer the questions below.

1. Is $7-5=5-7$ a true sentence? Is subtraction of whole numbers a commutative operation? Give other examples to verify your answer.

2. Is $(8-5)-3=8-(5-3)$ a true sentence? Is subtraction of whole numbers an associative operation? Give other examples to verify your answer.

3. Is $0-5=5$ a true sentence? Is $9-0=9$ a true sentence? What must zero be, the minuend or the subtrahend, in order to be called the identity number of subtraction?

Tell the subtraction sentence you would use to solve each open sentence below.

a	b
4. $9+a=17$	$x+14=27$
5. $m+22=28$	$27+p=43$
6. $16+t=37$	$y+19=38$

Tell the addition sentence you would use to solve each open sentence below.

a	b
7. $b-15=7$	$17-n=8$
8. $c-9=11$	$99-r=73$
9. $s-7=13$	$84-w=69$

Written Use the set of whole numbers as the replacement set to solve the following open sentences.

1-6. Copy the open sentences in *Oral* 4-9. Solve each open sentence.

Copy and solve each open sentence below.

a	b
7. $(7-4)+4=c$	$(8+5)-5=d$
8. $r=(17+9)-9$	$m=(26-17)+17$
9. $(13-6)+6=k$	$(79+21)-21=r$
10. $(9-8)+a=9$	$16=(16+b)-5$
11. $t=(32+7)-7$	$(21-r)+r=21$

Can you do this? The whole number 8 can be expressed as the sum of two whole numbers in 9 different ways. State them.

Tell why Any whole number n can be expressed as the sum of two whole numbers in $n+1$ different ways. Give an argument for the truth of this statement.

Tell how Tell how inverse operations are used to solve the following open sentence.

$$\begin{aligned}(7+n)-6 &= 18 \\ 7+n &= 18+6 \\ 7+n &= 24 \\ n &= 24-7 \\ n &= 17\end{aligned}$$

Techniques of Subtraction

You can find the number named by $746 - 234$ by thinking about the expanded notation for each number.

$746 - 234$	$\begin{array}{r} 700 + 40 + 6 \\ - (200 + 30 + 4) \\ \hline 500 + 10 + 2 \end{array}$	$\begin{array}{r} 746 \\ - 234 \\ \hline 512 \end{array}$
-------------	--	---

Notice that subtraction is performed in each place-value position. If $746 - 234 = 512$, then $746 = 512 + 234$. Why? What operation can you use to check a subtraction result?

To find the number named by $854 - 326$, you cannot subtract in every place-value position until the minuend is renamed.

$854 - 326$	$\begin{array}{r} 800 + 50 + 4 \\ - (300 + 20 + 6) \\ \hline \end{array}$	$\begin{array}{r} 800 + 40 + 14 \\ - (300 + 20 + 6) \\ \hline 500 + 20 + 8 \end{array}$	$\begin{array}{r} 854 \\ - 326 \\ \hline 528 \end{array}$
-------------	---	---	---

Why is $800 + 50 + 4$ renamed as $800 + 40 + 14$? How is $500 + 20 + 8$ obtained? How can you check that 528 is the correct result?

In some subtraction examples it is necessary to think of several names for the minuend in order to be able to subtract in every place-value position. Consider finding the number named by $461 - 274$.

$\begin{array}{r} 400 + 60 + 1 \\ - (200 + 70 + 4) \\ \hline ? \end{array}$	$\begin{array}{r} 400 + 50 + 11 \\ - (200 + 70 + 4) \\ \hline ? + 7 \end{array}$	$\begin{array}{r} 300 + 150 + 11 \\ - (200 + 70 + 4) \\ \hline 100 + 80 + 7 \end{array}$	$\begin{array}{r} 461 \\ - 274 \\ \hline 187 \end{array}$
---	--	--	---

Why must $400 + 50 + 11$ be renamed as $300 + 150 + 11$? How is this renaming done? How can you check that 187 is the correct result?

In which of the following examples must you think of renaming the minuend in order to do the subtraction?

$$\begin{array}{r} 347 \\ - 206 \\ \hline \end{array}$$

$$\begin{array}{r} 347 \\ - 209 \\ \hline \end{array}$$

$$\begin{array}{r} 347 \\ - 252 \\ \hline \end{array}$$

$$\begin{array}{r} 347 \\ - 258 \\ \hline \end{array}$$

Oral How would you rename each minuend below in order to find the difference?

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\begin{array}{r} 43 \\ -27 \\ \hline \end{array}$	$\begin{array}{r} 864 \\ -358 \\ \hline \end{array}$	$\begin{array}{r} 7851 \\ -378 \\ \hline \end{array}$
2.	$\begin{array}{r} 495 \\ -249 \\ \hline \end{array}$	$\begin{array}{r} 3000 \\ -1461 \\ \hline \end{array}$	$\begin{array}{r} 32061 \\ -14372 \\ \hline \end{array}$

Written Copy. Find each difference. Check each result by using addition.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\begin{array}{r} 65 \\ -43 \\ \hline \end{array}$	$\begin{array}{r} 7879 \\ -5384 \\ \hline \end{array}$	$\begin{array}{r} 85853 \\ -57906 \\ \hline \end{array}$
2.	$\begin{array}{r} 785 \\ -262 \\ \hline \end{array}$	$\begin{array}{r} 8341 \\ -3051 \\ \hline \end{array}$	$\begin{array}{r} 75787 \\ -26909 \\ \hline \end{array}$
3.	$\begin{array}{r} 945 \\ -416 \\ \hline \end{array}$	$\begin{array}{r} 7648 \\ -4926 \\ \hline \end{array}$	$\begin{array}{r} 69557 \\ -42589 \\ \hline \end{array}$
4.	$\begin{array}{r} 247 \\ -36 \\ \hline \end{array}$	$\begin{array}{r} 7000 \\ -1654 \\ \hline \end{array}$	$\begin{array}{r} 52691 \\ -24808 \\ \hline \end{array}$
5.	$\begin{array}{r} 957 \\ -316 \\ \hline \end{array}$	$\begin{array}{r} 6001 \\ -1387 \\ \hline \end{array}$	$\begin{array}{r} 30604 \\ -13365 \\ \hline \end{array}$
6.	$\begin{array}{r} 683 \\ -257 \\ \hline \end{array}$	$\begin{array}{r} 4006 \\ -2439 \\ \hline \end{array}$	$\begin{array}{r} 42981 \\ -27819 \\ \hline \end{array}$
7.	$\begin{array}{r} 584 \\ -196 \\ \hline \end{array}$	$\begin{array}{r} 5853 \\ -2906 \\ \hline \end{array}$	$\begin{array}{r} 70003 \\ -32840 \\ \hline \end{array}$
8.	$\begin{array}{r} 623 \\ -285 \\ \hline \end{array}$	$\begin{array}{r} 6071 \\ -5391 \\ \hline \end{array}$	$\begin{array}{r} 90502 \\ -60827 \\ \hline \end{array}$

Another way Mental computation can be done rapidly and easily by thinking about subtraction in different ways.

To subtract 19 from 73, you might think as follows:

19 is 1 less than 20, so I'll subtract 20 and then add 1 to the result.

$$\begin{aligned} 73 - 19 &= (73 - 20) + 1 \\ &= 53 + 1 \text{ or } 54 \end{aligned}$$

To subtract 48 from 91, you might think as follows:

48 is 2 less than 50, so I'll subtract 50 and then add 2 to the result.

$$\begin{aligned} 91 - 48 &= (91 - 50) + 2 \\ &= 41 + 2 \text{ or } 43 \end{aligned}$$

To subtract 27 from 63, you might think as follows:

27 = 30 - 3, so I'll subtract 30 from 63 and then add 3.

$$\begin{aligned} 63 - 27 &= (63 - 30) + 3 \\ &= 33 + 3 \text{ or } 36 \end{aligned}$$

Use any of the above methods to mentally find each difference below.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	54 - 19	83 - 17	100 - 32
2.	76 - 28	45 - 29	120 - 56
3.	61 - 25	32 - 18	271 - 47

Multiplication of Whole Numbers

Whole numbers = $\{0, 1, 2, 3, 4, \dots, 10, 11, 12, 13, 14, \dots\}$

$$3 \times 4 = 12$$

Is 3 a member of the set of whole numbers? Is 4 a member of the set of whole numbers? Is the product of 3 and 4 also a member of the set of whole numbers? Could 3×4 name some other number, like 7 or $\frac{3}{4}$, that is different than 12?

Multiplication is an operation on two numbers of a set resulting in a third number of the set. The numbers being multiplied are called **factors**. The resulting number is called the **product**.

Since the \times symbol for multiplication might be confused with the variable x , let us agree on some other notations for multiplication. Each of the following means the same as 3×4 .

$$3 \cdot 4$$

$$(3)(4)$$

$$3(4)$$

$$(3)4$$

When using a dot to indicate multiplication, be sure to raise the dot so as not to mistake it for a decimal point.

Recall that $3 \times 5 = 5 + 5 + 5$. By thinking of multiplication as repeated addition and by using only the results and properties of addition, you can prove that $3 \times 5 = 5 \times 3$. That is, to prove the multiplication statement $3 \times 5 = 5 \times 3$, let us agree *not* to use any results of multiplication.

$$3 \times 5 = 5 + 5 + 5$$

$$= (3 + 2) + (3 + 2) + (3 + 2)$$

$$= (3 + 3 + 3) + (2 + 2 + 2)$$

$$= (3 + 3 + 3) + [2 + (1 + 1) + 2]$$

$$= (3 + 3 + 3) + [(2 + 1) + (1 + 2)]$$

$$= 3 + 3 + 3 + 3 + 3$$

$$= 5 \times 3$$

Multiplication as repeated addition

Another numeral for 5

Addition is commutative and associative.

Another numeral for 2

Addition is associative.

Simplest name for $2 + 1$ and $1 + 2$

Repeated addition as multiplication

Commutative Property of Multiplication: If a and b represent whole numbers, then

$$a \times b = b \times a \text{ or } ab = ba.$$

You can also prove that $(2 \times 3) \times 4 = 2 \times (3 \times 4)$.

$$(2 \times 3) \times 4 = (3 + 3) \times 4$$

$$= 4 \times (3 + 3)$$

$$= (3 + 3) + (3 + 3) + (3 + 3) + (3 + 3)$$

$$= (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3)$$

$$= (4 \times 3) + (4 \times 3)$$

$$= (3 \times 4) + (3 \times 4)$$

$$= 2 \times (3 \times 4)$$

Why can 2×3 be stated as $3 + 3$?

Multiplication is commutative.

Why?

Why?

Why?

Multiplication is commutative.

Why?

Associative Property of Multiplication: If a , b , and c represent whole numbers, then

$$(a \times b) \times c = a \times (b \times c) \text{ or } (ab)c = a(bc).$$

Oral Tell which property of multiplication is illustrated by each sentence below.

1. $52 \times 123 = 123 \times 52$

2. $(17 \times 9) \times 34 = 17 \times (9 \times 34)$

3. $(26a)r = 26(ar)$

4. $2365t = t \times 2365$

Tell which numeral should replace each variable so that the resulting sentence is true.

a

b

5. $9a = 3 \times 9$ $(4 \times 2) \times 3 = 4(2b)$

6. $7 \times 5 = r \times 7$ $(7 \times 5)r = 7(5r)$

7. $n \times 3 = 3 \times 1$ $c(3 \times 5) = (6 \times 3) \times 5$

8. $6 \times 11 = 11s$ $(9k) \times 13 = 9(7 \times 13)$

9. $12a = a \times 12$ $3m = 15 + 15 + 15$

Written Find the simplest numeral for the number named by each expression below.

a

b

1. 7×9

$8 \times (5 \times 2)$

2. 13×4

$(8 \times 5) \times 2$

3. 6×17

$(14 \times 5) \times 3$

4. 23×5

$14 \times (5 \times 3)$

5. 3×56

$8(21 \times 2)$

6. $(7)(24)$

$(15 \times 8) \times 6$

7. $17(8)$

$3(17 \times 4)$

Can you do this? Prove each of the following statements without using results of multiplication.

1. $2 \times 4 = 4 \times 2$

2. $(3 \times 5) \times 2 = 3 \times (5 \times 2)$

Using the Properties of Multiplication

×	0	1	2	3	4	5	6	7	8	9
0	0									
1	0	1								
2	0	2	4							
3	0	3	6	9						
4	0	4	8	12	16					
5	0	5	10	15	20	25				
6	0	6	12	18	24	30	36			
7	0	7	14	21	28	35	42	49		
8	0	8	16	24	32	40	48	56	64	
9	0	9	18	27	36	45	54	63	72	81

If you know how to find the products named in this table, and if you use the properties of multiplication, then finding any product becomes quite an easy task.

Why is it a waste of effort to memorize the entries that could be written in the blank part of the table?

Notice that $1 \times 0 = 0$, that $2 \times 0 = 0$, and so on in the 0-column. If one of the factors is zero, the product is what number? If a product is zero, what can you conclude about the factors?

If n represents any whole number, then $n \times 0 = 0 = 0 \times n$.

If a product is zero, then at least one of the factors is zero.

Notice also that $1 \times 1 = 1$, that $2 \times 1 = 2$, that $3 \times 1 = 3$, and so on in the 1-column. If one of two factors is the number one, what can you conclude about the product?

The number one is called the **identity number of multiplication**. That is, if n represents any whole number, then

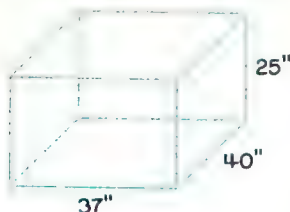
$$n \times 1 = n = 1 \times n.$$

By using the associative property of multiplication and multiples of powers of ten, you can easily find other products.

$$\begin{aligned} 5 \times 600 &= 5 \times (6 \times 100) \\ &= (5 \times 6) \times 100 \\ &= 30 \times 100 \text{ or } 3000 \end{aligned}$$

$$\begin{aligned} (8 \times 47) \times 125 &= (47 \times 8) \times 125 \\ &= 47 \times (8 \times 125) \\ &= 47 \times 1000 \text{ or } 47000 \end{aligned}$$

Oral Some pupils were required to find the volume measure of the rectangular solid shown below.



Tom's work: $(37 \times 25) \times 40$

$$\begin{array}{r} 37 \\ \times 25 \\ \hline 185 \\ 740 \\ \hline 925 \end{array} \qquad \begin{array}{r} 925 \\ \times 40 \\ \hline 37000 \end{array}$$

Pat's work: $(37 \times 25) \times 40$

$$\begin{aligned} (37 \times 25) \times 40 &= 37 \times (25 \times 40) \\ &= 37 \times 1000 \\ &= 37000 \end{aligned}$$

Answer the following.

1. Explain how Tom found the volume measure.

2. Explain how Pat found the volume measure.

3. Which person did less work?

4. How would you solve the open sentence $(27 \times 250) \times 4 = n$ in the easiest way?

5. Grouping symbols are not used in the open sentence $7 \times 125 \times 4 = t$. How would you solve the open sentence in the easiest way?

Written Write the letter of the most correct ending for each sentence.

1. If $m \times n = 0$, then (a. $m = 0$, b. $n = 0$, c. either m or n or both are equal to zero).

2. If $37 \times n = 37$, then (a. $n = 0$, b. $n = 1$, c. n can represent any whole number).

3. If $k \times 0 = y$, then (a. $y = 0$, b. $k = 0$, c. $k = 1$ and $y = 0$).

Do not copy the following. Just write the simplest numeral for each product.

	<i>a</i>	<i>b</i>	<i>c</i>
4.	4×70	4×700	40×700
5.	9×80	90×80	90×8000

Copy. Use the properties of multiplication to find each product in the easiest way.

	<i>a</i>	<i>b</i>
6.	$2 \times 9 \times 5$	$25 \times 23 \times 8$
7.	$8 \times 19 \times 25$	$20 \times 16 \times 50$
8.	$5 \times 20 \times 43$	$2 \times 24 \times 500$
9.	$2 \times 38 \times 50$	$25 \times 63 \times 40$
10.	$25 \times 83 \times 4$	$8 \times 17 \times 125$
11.	$500 \times 17 \times 2$	$25 \times 67 \times 4$
12.	$25 \times 31 \times 4$	$50 \times 84 \times 20$

Distributive Property of Multiplication over Addition



A factory charges \$17 for a box of merchandise. It distributed merchandise to two of its outlets as follows:

Outlet A: 10 boxes

Outlet B: 3 boxes

What is the total value of all the merchandise distributed?

You might think:

There are $10+3$ boxes in all. Each box of merchandise costs \$17, so the total value of all the merchandise is $17(10+3)$ dollars.

You might think:

The value of the merchandise to Outlet A is 17×10 and to Outlet B is 17×3 . The total value is $(17 \times 10) + (17 \times 3)$ dollars.

Since the total value should be the same, regardless of how it is computed, the following statement must be true.

$$\begin{aligned} 17(10+3) &= (17 \times 10) + (17 \times 3) \\ &= 170 + 51 \\ &= 221 \end{aligned}$$

This example illustrates a property that links the operations of addition and multiplication.

Distributive Property of Multiplication over Addition: If a , b , and c represent whole numbers, then

$$a(b+c) = (a \times b) + (a \times c) \text{ or } a(b+c) = ab + ac.$$

To apply this property, rename one of the factors as a sum and distribute multiplication over addition.

$$\begin{aligned} 8 \times 47 &= 8 \times (40+7) \\ &= (8 \times 40) + (8 \times 7) \\ &= 320 + 56 \text{ or } 376 \end{aligned}$$

$$\begin{aligned} 9 \times 236 &= 9 \times (200+30+6) \\ &= (9 \times 200) + (9 \times 30) + (9 \times 6) \\ &= 1800 + 270 + 54 \text{ or } 2124 \end{aligned}$$

Since multiplication is commutative, you can easily show another pattern of the distributive property of multiplication over addition.

$$a(b+c) = ab+ac$$

Now commute the factors in each product.

$$(b+c)a = ba+ca$$

To compute 15×24 , name either factor as a sum and distribute multiplication over addition in the most convenient way.

$$\begin{aligned} 15 \times 24 &= 15 \times (20+4) \\ &= (15 \times 20) + (15 \times 4) \\ &= 300 + 60 \text{ or } 360 \end{aligned}$$

$$\begin{aligned} 15 \times 24 &= (10+5) \times 24 \\ &= (10 \times 24) + (5 \times 24) \\ &= 240 + 120 \text{ or } 360 \end{aligned}$$

Oral Answer the questions below.

1. To compute 15×24 , why is 15 renamed as $10+5$ instead of as $9+6$, or $7+8$, or $12+3$?

2. To compute 8×324 , which factor would you name as a sum? How would you name it as a sum?

3. To find the simplest numeral for 23×14 , which factor would you rename as a sum? Could you rename either factor as a sum?

4. How many operations are involved when speaking of a commutative property? Of an associative property? Of a distributive property?

5. How would you state the distributive property of multiplication over addition in your own words?

Written Use the distributive property of multiplication over addition to find the simplest numeral for each product named below.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	3×46	71×3	6×291
2.	6×28	68×4	7×199
3.	7×73	57×3	4×272
4.	2×56	47×6	3×165
5.	8×75	28×7	9×254
6.	9×47	37×8	8×297
7.	4×89	95×5	9×781
8.	7×29	75×9	5×543
9.	5×86	57×8	7×236

The Multiplication Algorithm

By using the distributive property of multiplication over addition, you can develop an algorithm for multiplication. An algorithm is simply a convenient arrangement of numerals designed to obtain the simplest numeral for the result of an operation in the easiest possible way.

$$\begin{aligned}
 5 \times 27 &= 5(20+7) \\
 &= (5 \times 20) + (5 \times 7) \\
 &= 100 + 35
 \end{aligned}$$

$$\begin{array}{r}
 27 \\
 \times 5 \\
 \hline
 135
 \end{array}$$

You can shorten the algorithm by doing some of the computation mentally.

$$\begin{array}{r}
 \overset{3}{27} \\
 \times 5 \\
 \hline
 5
 \end{array}$$

$$\begin{array}{r}
 \overset{3}{27} \\
 \times 5 \\
 \hline
 135
 \end{array}$$

$$\begin{array}{r}
 27 \\
 \times 5 \\
 \hline
 135
 \end{array}$$

Think: $7 \times 5 = 35 = 30 + 5$ $(20 \times 5) + 30 = 100 + 30$

To find the product of greater numbers, the place value of each digit can be shown on a grid.

H	T	O
2	7	
×	3	5
<hr/>		
3	5	5×7
1	0	5×20
2	1	30×7
6	0	30×20
9	4	5

H	T	O
2	7	
×	3	5
<hr/>		
1	3	5
8	1	0
9	4	5

$$\begin{array}{r}
 27 \\
 \times 35 \\
 \hline
 135 \\
 810 \\
 \hline
 945
 \end{array}$$

The final form of the above algorithm can be written as shown at the left below.

$$\begin{array}{r}
 27 \\
 \times 35 \\
 \hline
 135 \\
 81 \\
 \hline
 945
 \end{array}$$

The 0 can be omitted here since $5 + 0 = 5$.

Which property of addition does this illustrate?

It is certainly not wrong to write this 0, but it is convenient to omit it. Algorithms are designed for convenience.

Oral Explain how each of the colored numerals are obtained in the example below.

a	b
$\begin{array}{r} 491 \\ \times 9 \\ \hline 9 \\ 810 \\ 3600 \\ \hline 4419 \end{array}$	$\begin{array}{r} 621 \\ \times 324 \\ \hline 2484 \\ 1242 \\ 1863 \\ \hline 201204 \end{array}$

Written Copy. Find the simplest numeral for each product.

	a	b	c
1.	$\begin{array}{r} 57 \\ \times 9 \\ \hline \end{array}$	$\begin{array}{r} 349 \\ \times 3 \\ \hline \end{array}$	$\begin{array}{r} 7468 \\ \times 5 \\ \hline \end{array}$
2.	$\begin{array}{r} 37 \\ \times 26 \\ \hline \end{array}$	$\begin{array}{r} 469 \\ \times 20 \\ \hline \end{array}$	$\begin{array}{r} 3406 \\ \times 32 \\ \hline \end{array}$
3.	$\begin{array}{r} 87 \\ \times 56 \\ \hline \end{array}$	$\begin{array}{r} 537 \\ \times 44 \\ \hline \end{array}$	$\begin{array}{r} 2078 \\ \times 26 \\ \hline \end{array}$
4.	$\begin{array}{r} 95 \\ \times 48 \\ \hline \end{array}$	$\begin{array}{r} 521 \\ \times 423 \\ \hline \end{array}$	$\begin{array}{r} 5437 \\ \times 354 \\ \hline \end{array}$
5.	$\begin{array}{r} 88 \\ \times 37 \\ \hline \end{array}$	$\begin{array}{r} 708 \\ \times 309 \\ \hline \end{array}$	$\begin{array}{r} 6342 \\ \times 406 \\ \hline \end{array}$
6.	$\begin{array}{r} 76 \\ \times 40 \\ \hline \end{array}$	$\begin{array}{r} 645 \\ \times 440 \\ \hline \end{array}$	$\begin{array}{r} 8057 \\ \times 4242 \\ \hline \end{array}$
7.	$\begin{array}{r} 27 \\ \times 43 \\ \hline \end{array}$	$\begin{array}{r} 305 \\ \times 440 \\ \hline \end{array}$	$\begin{array}{r} 3535 \\ \times 4242 \\ \hline \end{array}$
8.	$\begin{array}{r} 84 \\ \times 59 \\ \hline \end{array}$	$\begin{array}{r} 748 \\ \times 321 \\ \hline \end{array}$	$\begin{array}{r} 4537 \\ \times 1234 \\ \hline \end{array}$

Write an open sentence for each problem below. Solve the open sentence. Write an answer for the problem.

9. Twelve new automobiles are loaded on railroad cars. If each automobile weighs 3450 pounds, what is their total weight?

10. A pilot made 156 flights last year. The average distance per flight was 927 miles. How many miles did he fly last year?

11. A company employs 207 people and has a 7-hour working day. How many man-hours are worked each day by the employees of this company?

12. It is 17 miles by car from Mr. Jones' home to his office. How many miles does he drive in making 23 round trips to his home and his office?

13. A space ship in orbit traveled at an average speed of 17,000 miles per hour. How many miles did it travel in 24 hours?

Can you do this? Replace each with a numeral to make a correctly worked multiplication example.

$$\begin{array}{r} 7 \\ \times 703 \\ \hline 1 \\ 7 9 \\ \times 91 \\ \hline 91 \end{array}$$

The Inverse Operation of Multiplication

Just as man has created subtraction as the inverse operation of addition, he has created an inverse operation of multiplication. To undo a multiplication result, we have an operation called division. Division is related to multiplication as follows.

$$\text{If } 4 \times 7 = 28, \text{ then } 28 \div 7 = 4.$$

$$\text{If } 7 \times 4 = 28, \text{ then } 28 \div 4 = 7.$$

How is “ $\times 7$ ” undone above? How is “ $\times 4$ ” undone?

Let a , b , and c represent whole numbers, but $a \neq 0$ and $b \neq 0$.

$$\text{If } ab = c, \text{ then } a = c \div b \text{ or } a = \frac{c}{b}$$

$$\text{and } b = c \div a \text{ or } b = \frac{c}{a}.$$

Since division is the inverse operation of multiplication, you should expect that multiplication is the inverse operation of division.

$$\text{If } 36 \div x = 9, \text{ then } 9x = 36.$$

$$\text{If } k \div 3 = 7, \text{ then } 7 \times 3 = k.$$

What is the inverse of *dividing by x* ? Of *dividing by 3*?

Suppose you start with some natural number a and multiply it by a natural number b . What could you do to this product to obtain the number a with which you started?

Suppose you start with some whole number a and divide it by a natural number b . What could you do to this quotient to obtain the number a with which you started?

If a and b represent whole numbers, but $b \neq 0$,

$$\text{then } (a \times b) \div b = a \text{ or } \frac{ab}{b} = a$$

$$\text{and } (a \div b) \times b = a \text{ or } \frac{a}{b}(b) = a.$$

The idea of inverse operations can be very useful in solving certain open sentences. In the following examples, let the replacement set be the set of whole numbers.

$$3x=12$$

$$x=12\div 3$$

$$x=4$$

Solution set is $\{4\}$.

$$k\div 16=3$$

$$k=3\times 16$$

$$k=48$$

Solution set is $\{48\}$.

$$\frac{28}{r}=4$$

$$28=4r$$

$$\frac{28}{4}=r$$

$$7=r$$

Solution set is $\{7\}$.

Explain how inverse operations are used to solve each open sentence above.

Oral Answer the questions below.

1. Is $8\div 2=2\div 8$ a true sentence? Is division of whole numbers a commutative operation?

2. Is $(12\div 4)\div 2=12\div (4\div 2)$ a true sentence? Is division of whole numbers an associative operation?

3. Is $9\div 1=9$ a true sentence? Is $1\div 9=9$ a true sentence? What must 1 be, the dividend or the divisor, in order to be called the identity number of division?

Tell the division or multiplication sentence you would use to solve each open sentence below.

a

4. $7t=42$

5. $n\times 15=75$

6. $\frac{m}{8}=1$

b

$s\times 9=54$

$12k=144$

$4=\frac{n}{17}$

Written Use the set of whole numbers as the replacement set to solve the following open sentences.

a

1. $7x=98$

2. $n\times 15=90$

3. $72\div k=8$

4. $25\times s=3$

5. $(7\times 6)\div c=7$

6. $x=(15\times 9)\div 9$

7. $125=k\times 5$

8. $\frac{r}{13}=7$

b

$a\times 11=77$

$4t=128$

$m\div 6=16$

$11=b\div 12$

$(12\div n)\times 4=12$

$18=(a\div 6)\times 6$

$1000=100n$

$\frac{c}{81}=27$

Tell why In $3x=10$, the product is greater than the named factor. However, there is no solution in the set of whole numbers for this open sentence. Why?

Zero in Division

You have learned that zero has some special properties in addition, subtraction, and multiplication. Let us investigate zero in division.

$0 \div 5 = n$ *The dividend is zero and the divisor is not zero.*

If $0 \div 5 = n$, then $0 = n \times 5$. Why?

What is the solution set for $0 = n \times 5$? For $0 \div 5 = n$?

If zero is divided by any natural number, the quotient is zero.

$7 \div 0 = n$ *The divisor is zero.*

If $7 \div 0 = n$, then $7 = n \times 0$. Why?

What is the solution set for $7 = 0 \times n$? For $7 \div 0 = n$?

If $0 \div 0 = n$, then $0 = 0 \times n$. In this case the solution set is the set of all numbers and there is no single quotient.

Division by zero is meaningless. Zero is not used as a divisor.

Oral Tell which of the following are meaningless expressions.

- | a | b | c |
|---------------|----------------|-------------------|
| 1. $0 \div 8$ | $\frac{0}{13}$ | $18 \div (6 - 6)$ |
| 2. $3 \div 0$ | $\frac{17}{0}$ | $(9 - 9) \div 15$ |

Answer the following questions.

3. What number is named by 4×0 ? By $0 \div 4$? By $4 \div 0$?

4. If a quotient is zero, what can you conclude about the dividend and the divisor?

Written Copy. Solve each equation.

- | a | b |
|-------------------------|----------------------------|
| 1. $0 \div 11 = n$ | $32 \div (2 + 6) = n$ |
| 2. $n \div 8 = 0$ | $24 \div (8 \times 3) = n$ |
| 3. $56 \div n = 8$ | $(17 - 5) \div n = 2$ |
| 4. $\frac{n}{12} = 168$ | $\frac{n}{5 \times 3} = 0$ |
| 5. $0 \div n = 0$ | $(11 - n) \div 6 = 0$ |
| 6. $110 \div n = 10$ | $(32 - 32) \div 7 = n$ |
| 7. $n \div 117 = 0$ | $(n - 15) \div 3 = 0$ |

Divisibility of Whole Numbers

Since $3 \times 4 = 12$, you can conclude the following:

3 is a factor of 12.

12 is a multiple of 3.

4 is a factor of 12.

12 is a multiple of 4.

Since division is the inverse operation of multiplication, you can also conclude the following from $3 \times 4 = 12$.

$$12 \div 3 = 4$$

12 is divisible by 3.

$$12 \div 4 = 3$$

12 is divisible by 4.

Let a and b represent whole numbers, but $a \neq 0$.

If a is a factor of b , then b is divisible by a .

Think of solving $3n = 20$ by using the set of whole numbers as the replacement set. What is the solution set? Is 3 a factor of 20?

If 3 is not a factor of 20, then 20 is not divisible by 3. What is the greatest whole number less than 20 that is divisible by 3? How many times is 3 contained in that number? In the system of whole numbers, you can express this as:

$$(6 \times 3) + 2 = 20$$

or

$$20 \div 3 = 6 \text{ r}2$$

Oral Answer the questions below.

1. What is meant by *one number is divisible by another number*?

2. Which whole numbers less than 32 are divisible by 5?

3. What are the quotient and the remainder for $32 \div 5$?

Written Find the quotient and the remainder for each of the following.

a

b

c

1. $27 \div 5$

$48 \div 8$

$53 \div 9$

2. $19 \div 6$

$24 \div 7$

$26 \div 4$

3. $37 \div 9$

$37 \div 4$

$56 \div 7$

Using a Multiplication Table for Division

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Since division is the inverse of multiplication, you can solve some division sentences by using the multiplication table.

$$48 \div 6 = k$$

- Locate 48 in the 6-column.
- Locate the first numeral (at the left) in that row.

$$48 \div 6 = 8$$

Now think of solving $20 \div 3 = n$ in the same manner. Look for 20 in the 3-column. Since 20 does not appear in the 3-column, you know that 20 is not divisible by 3. However, the greatest number less than 20 that is divisible by 3 is named in that column. How can you locate the numeral for that number? How can you determine the quotient and remainder for $20 \div 3$?

Oral Answer the questions below.

- Do $6 \div 3 = 2$ and $6 \div 3 = 2r0$ mean the same thing?
- Can zero be a remainder?
- If the divisor is 3, what are the only possible remainders?
- If the divisor is 5, what are the only possible remainders?
- Why must the remainder always be less than the divisor?

Written Find the quotient and the remainder for each of the following.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$31 \div 7$	$42 \div 6$	$57 \div 9$
2.	$27 \div 4$	$39 \div 8$	$18 \div 5$
3.	$13 \div 2$	$23 \div 3$	$22 \div 4$
4.	$72 \div 8$	$77 \div 9$	$19 \div 3$

5–8. Write a sentence like $31 \div 7 = 4 r3$ and $31 = (7 \times 4) + 3$ for each of the expressions in *Written* 1–4.

Division Is Distributive over Addition

$\frac{8}{4}$ is another numeral for $8 \div 4$ or 2

$\frac{12}{4}$ is another numeral for $12 \div 4$ or 3

You already know how to add $\frac{8}{4}$ and $\frac{12}{4}$ as shown below.

$$\frac{8}{4} + \frac{12}{4} = \frac{8+12}{4} = \frac{20}{4}$$

You can reverse the order of the above equality sentences by using the symmetric property of equality. You can replace the fractions by division numerals. Then you can discover an important relationship between division and addition.

$$\frac{20}{4} = \frac{8+12}{4} = \frac{8}{4} + \frac{12}{4} = 2 + 3 = 5$$

$$20 \div 4 = (8+12) \div 4 = (8 \div 4) + (12 \div 4) = 2 + 3 = 5$$

Is the dividend or the divisor renamed as a sum in going from $20 \div 4$ to $(8+12) \div 4$? Why is 20 renamed as $8+12$ rather than $13+7$ or $10+10$? How does the above example resemble the distributive property of multiplication over addition?

Division can be distributed over addition if the dividend is renamed as a sum.

Oral Which of the following are usable ways of renaming the dividend when dividing 42 by 6? Why?

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$40+2$	$30+12$	$20+22$

2.	$24+18$	$32+10$	$36+6$
----	---------	---------	--------

3. Which of the addition numerals in *Oral* 1 and 2 is most convenient to use?

Written Find each quotient by renaming the dividend and distributing division over addition.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$52 \div 4$	$65 \div 5$	$63 \div 3$

2.	$60 \div 4$	$70 \div 5$	$96 \div 8$
----	-------------	-------------	-------------

3.	$72 \div 4$	$64 \div 4$	$42 \div 3$
----	-------------	-------------	-------------

4.	$76 \div 4$	$26 \div 2$	$80 \div 5$
----	-------------	-------------	-------------

The Division Algorithm

To solve $245 \div 7 = a$ and $672 \div 6 = n$, you can distribute division over addition as shown below.

$$\begin{aligned} 245 \div 7 &= (210 + 35) \div 7 \\ &= (210 \div 7) + (35 \div 7) \\ &= 30 + 5 \text{ or } 35 \end{aligned}$$

$$\begin{aligned} 672 \div 6 &= (600 + 60 + 12) \div 6 \\ &= (600 \div 6) + (60 \div 6) + (12 \div 6) \\ &= 100 + 10 + 2 \text{ or } 112 \end{aligned}$$

Another way to show the renaming and the division is given below.

$$\begin{array}{r} 30 + 5 = 35 \\ 7 \overline{)210 + 35} \end{array}$$

$$\begin{array}{r} 100 + 10 + 2 = 112 \\ 6 \overline{)600 + 60 + 12} \end{array}$$

Notice how the dividends are renamed in the examples above. Notice also that the quotient is first expressed in expanded notation. Such divisions are easy once the dividend is renamed in the most convenient form. So, the question that needs answering is "How does one find the most convenient form for renaming the dividend?"

You want the quotient to be a sum of *ones*, *tens*, *hundreds*, and so on. So rename the dividend as a sum of multiples of powers of ten and the divisor. You can round off the dividend to make the renaming easier.

$$6 \overline{)672}$$

Round off 672 to the nearest hundred.
700 or 7×100 or 7H

Now think: $6 \times __\text{H} \leq 7\text{H}$ or $7\text{H} \div 6 \geq __\text{H}$

What is the greatest whole number that $__\$ can represent so that either sentence above becomes true? Since $6 \times 1\text{H} = 6\text{H}$, you should think of 672 as $600 + 72$.

$$6 \overline{)600 + 72}$$

$$6 \overline{)600 + 72}$$

Now round off 72 to the nearest ten. 70 or 7×10 or 7T

Now think: $6 \times __\text{T} \leq 7\text{T}$ or $7\text{T} \div 6 \geq __\text{T}$

What is the greatest whole number that $__\$ can represent so that either sentence above becomes true? Since $6 \times 1\text{T} = 6\text{T}$, you should think of 72 as $60 + 12$.

$$6 \overline{)600 + 60 + 12}$$

Oral To rename the dividend in $528 \div 4$, you could use either $4 \times __H \leq 5H$ or $5H \div 4 \geq __H$ to obtain $(400 + 128) \div 4$.

1. What is the greatest whole number that $__$ can represent so that either open sentence above becomes true?

2. What is the hundreds digit of the quotient numeral?

3. Why is \leq used in the multiplication sentence?

4. Why is \geq used in the division sentence?

5. Why can either $4 \times __H \leq 5H$ or $5H \div 4 \geq __H$ be used to find the hundreds digit of the quotient? How are these two sentences related?

How would you round off each of the following to the nearest 100?

	<i>a</i>	<i>b</i>	<i>c</i>
6.	854	526	350
7.	682	450	747

How would you round off each of the following to the nearest 10?

	<i>a</i>	<i>b</i>	<i>c</i>
8.	57	35	47
9.	92	18	65

Written Rename each dividend as a sum. Then find the simplest numeral for each quotient.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$4 \overline{)488}$	$3 \overline{)369}$	$2 \overline{)284}$
2.	$7 \overline{)749}$	$8 \overline{)896}$	$6 \overline{)192}$
3.	$5 \overline{)365}$	$9 \overline{)198}$	$4 \overline{)480}$
4.	$6 \overline{)684}$	$5 \overline{)575}$	$2 \overline{)364}$
5.	$4 \overline{)492}$	$3 \overline{)426}$	$8 \overline{)272}$
6.	$2 \overline{)108}$	$7 \overline{)182}$	$7 \overline{)917}$
7.	$5 \overline{)615}$	$4 \overline{)852}$	$2 \overline{)628}$
8.	$6 \overline{)792}$	$8 \overline{)968}$	$9 \overline{)963}$
9.	$6 \overline{)144}$	$4 \overline{)208}$	$5 \overline{)585}$
10.	$3 \overline{)372}$	$7 \overline{)784}$	$6 \overline{)252}$

Tell how Without doing any computing, how can you tell that the quotient numeral for $3 \overline{)516}$ will contain 3 digits?

Without doing any computing, how can you tell that the quotient numeral for $6 \overline{)198}$ will contain only 2 digits?

Without doing any computing, how can you tell that the hundreds digit of the quotient numeral for $3 \overline{)732}$ will be 2?

The Division Algorithm

Renaming the dividend becomes clumsy and inconvenient, especially when greater numbers are used in division. Since the renaming is done according to place value, you can simply use the place value indicated by the dividend numeral.

$6 \overline{)672} \rightarrow 6 \overline{)600+60+12}$

$$\begin{array}{r} \text{H} \mid \text{T} \mid \text{O} \\ 1 \mid 1 \mid 2 \\ 6 \overline{)672} \\ \underline{600} \\ 72 \\ \underline{60} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

$7\text{H} \div 6 \geq \text{---H}$
 $6 \times 1\text{H} = 6\text{H}$
 $7\text{T} \div 6 \geq \text{---T}$
 $6 \times 1\text{T} = 6\text{T}$
 $12 \div 6 \geq \text{---}$

Why is it unnecessary to write the 0's here?

When greater numbers are used in division, you can round off both the dividend and the divisor to estimate each digit of the quotient numeral.

$$\begin{array}{r} \text{H} \mid \text{T} \mid \text{O} \\ 3 \mid \\ 32 \overline{)864} \\ \underline{960} \end{array}$$

Round off 32 to 3T. Round off 864 to 9H.
 $9\text{H} \div 3\text{T} \geq \text{---T}$
 $32 \times 3\text{T} = 960$

Since $960 > 864$, what can you conclude about 3 as the tens digit of the quotient? In this case, try the next lesser number of tens.

$$\begin{array}{r} \text{H} \mid \text{T} \mid \text{O} \\ 2 \mid \\ 32 \overline{)864} \\ \underline{64} \\ 224 \end{array}$$

Since 3T is too great, try 2T.

$$\begin{array}{r} \text{H} \mid \text{T} \mid \text{O} \\ 2 \mid 7 \mid \\ 32 \overline{)864} \\ \underline{64} \\ 224 \\ \underline{224} \\ 0 \end{array}$$

Round off 224 to 2H or 22T.
 $22\text{T} \div 3\text{T} \geq \text{---}$
 $32 \times 7 = 224$

What is the remainder in this division example?

It might be the case that your estimate of a digit in the quotient is not great enough. Explain where this occurs in the example below.

Th	H	T	O
56	7	3	46
	1		
	56		
	17		

→

Th	H	T	O
56	7	3	46
	1	2	
	56		
	17	4	
	11	2	
	6	2	

→

Th	H	T	O
56	7	3	46
	1	3	
	56		
	17	4	
	16	8	
	6	8	

→

Th	H	T	O
56	7	3	46
	1	3	1
	56		
	17	4	
	16	8	
		66	
		56	
		10	

7Th ÷ 6T ≥ ___H

17H ÷ 6T ≥ ___T

6T ÷ 6T ≥ ___

What is the remainder in this division example?

Oral Tell how you would round off the dividend and the divisor to estimate the hundreds digit of each quotient below.

- | <i>a</i> | <i>b</i> | <i>c</i> |
|-----------------------|----------------------|----------------------|
| 1. $5\overline{)621}$ | $42\overline{)2536}$ | $13\overline{)1755}$ |
| 2. $7\overline{)805}$ | $38\overline{)1462}$ | $54\overline{)3609}$ |
| 3. $4\overline{)936}$ | $62\overline{)2247}$ | $82\overline{)4237}$ |

Tell how you would estimate the tens digit of each quotient below.

- | <i>a</i> | <i>b</i> | <i>c</i> |
|--|---|--|
| 4. $26\overline{)364}$ | $54\overline{)1612}$ | $38\overline{)798}$ |
| 5. $7\overline{)1155}$
$\frac{7}{45}$ | $13\overline{)2942}$
$\frac{26}{34}$ | $34\overline{)4488}$
$\frac{34}{108}$ |

Written Copy. Find each quotient and remainder.

- | <i>a</i> | <i>b</i> | <i>c</i> |
|------------------------|----------------------|----------------------|
| 1. $7\overline{)2674}$ | $18\overline{)7257}$ | $37\overline{)3000}$ |
| 2. $8\overline{)5464}$ | $15\overline{)6109}$ | $67\overline{)7175}$ |
| 3. $8\overline{)3216}$ | $37\overline{)7955}$ | $23\overline{)5897}$ |
| 4. $3\overline{)457}$ | $42\overline{)9136}$ | $35\overline{)7350}$ |
| 5. $5\overline{)614}$ | $25\overline{)5760}$ | $29\overline{)9454}$ |
| 6. $6\overline{)385}$ | $54\overline{)3080}$ | $31\overline{)3514}$ |
| 7. $4\overline{)2736}$ | $93\overline{)9393}$ | $22\overline{)6798}$ |
| 8. $9\overline{)4329}$ | $37\overline{)9102}$ | $41\overline{)1298}$ |
| 9. $7\overline{)2741}$ | $64\overline{)8465}$ | $57\overline{)2384}$ |

Greater Numbers in Division

The process of rounding off both the dividend and the divisor to estimate digits of the quotient numeral can be extended when greater numbers are used in division. Further, you need not use the grid since any positional numeral automatically indicates place value.

Think of solving the equation $5047 \div 374 = n$. Since $374 \times 100 = 37400$, how do you know that there will be no hundreds digit in the quotient numeral? Since $374 \times 10 = 3740$, how do you know that there will be a tens digit in the quotient numeral? How many digits will the quotient numeral contain?

Explain how the quotient and the remainder are found in the following example.

$\begin{array}{r} 1 \\ 374 \overline{) 5047} \\ \underline{374} \\ 130 \end{array}$	→	$\begin{array}{r} 13 \\ 374 \overline{) 5047} \\ \underline{374} \\ 1307 \\ \underline{1122} \\ 185 \end{array}$	→	$\begin{array}{r} 13 \text{ r}185 \\ 374 \overline{) 5047} \\ \underline{374} \\ 1307 \\ \underline{1122} \\ 185 \end{array}$
5Th \div 4H \geq ___T		13H \div 4H \geq ___		

Is the final remainder less than the divisor? Notice how the remainder is indicated.

Explain how the quotient and the remainder are found in the following examples.

$\begin{array}{r} 1176 \text{ r}27 \\ 57 \overline{) 67059} \\ \underline{57} \\ 100 \\ \underline{57} \\ 435 \\ \underline{399} \\ 369 \\ \underline{342} \\ 27 \end{array}$	<p>7TTh \div 6T \geq ___Th</p> <p>10Th \div 6T \geq ___H</p> <p>43H \div 6T \geq ___T</p> <p>37T \div 6T \geq ___</p>	$\begin{array}{r} 120 \text{ r}334 \\ 478 \overline{) 57694} \\ \underline{478} \\ 989 \\ \underline{956} \\ 334 \end{array}$	<p>6TTh \div 5H \geq ___H</p> <p>10Th \div 5H \geq ___T</p> <p>3H \div 5H \geq ___</p>
---	---	---	--

$$\begin{array}{r} 27 \\ 26 \overline{) 702} \\ \underline{52} \\ 182 \\ \underline{182} \\ 0 \end{array}$$
 $18T \div 3T > \underline{\quad}$

9. If the remainder were not zero, how would you record it?

	a	b	c
1.	$24 \overline{)982}$	$23 \overline{)4425}$	$617 \overline{)5907}$

2. $26 \overline{)757}$ $62 \overline{)5151}$ $725 \overline{)7408}$

3. $16 \overline{)781}$ $54 \overline{)5143}$ $427 \overline{)8129}$

4. $31 \overline{)907}$ $25 \overline{)3732}$ $825 \overline{)9284}$

5. $68 \overline{)504}$ $51 \overline{)7124}$ $834 \overline{)76521}$

6. $19 \overline{)726}$ $22 \overline{)4972}$ $729 \overline{)69529}$

7. $45 \overline{)765}$ $16 \overline{)9746}$ $615 \overline{)48765}$

8. $34 \overline{)884}$ $37 \overline{)5727}$ $303 \overline{)16057}$

9. $53 \overline{)615}$ $27 \overline{)5699}$ $721 \overline{)89900}$

10. $63 \overline{)945}$ $38 \overline{)3838}$ $507 \overline{)35764}$

11. $78 \overline{)702}$ $77 \overline{)3703}$ $428 \overline{)65000}$

12. $33 \overline{)792}$ $42 \overline{)9624}$ $213 \overline{)16075}$

13. $47 \overline{)564}$ $62 \overline{)8247}$ $413 \overline{)27854}$

$$\begin{array}{r} a \\ 27 \overline{) 522} \\ \underline{54} \\ 32 \\ \underline{27} \\ 52 \\ \underline{54} \\ 25 \end{array}$$

$$\begin{array}{r} b \\ 7 \\ \hline 64293 \\ 69 \\ \hline 339 \\ 261 \\ \hline 73 \\ \hline 0 \end{array}$$

Divisibility by 2 and by 5

The following is the set of even numbers.

$\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, \dots\}$

Is 2 a factor of every even number? Is every even number divisible by 2?

Observe the ones digit in the numerals that name even numbers. What pattern can you discover? By examining a base-ten numeral for a whole number, how can you tell whether or not it names an even number?

By counting by fives, you can form the following set of numbers.

$\{0, 5, 10, 15, 20, 25, 30, 35, 40, 45, \dots\}$

Is 5 a factor of each of those numbers? Is each of those numbers divisible by 5?

Observe the ones digit in the numerals that name numbers that are divisible by 5. What pattern do you discover? By examining a base-ten numeral for a whole number, how can you tell whether or not it names a number that is divisible by 5?

Think of finding the simplest numeral for the product $2 \times 5 \times a$ when a represents a whole number.

$$2 \times 5 \times 0 = 0$$

$$2 \times 5 \times 2 = 20$$

$$2 \times 5 \times 4 = 40$$

$$2 \times 5 \times 1 = 10$$

$$2 \times 5 \times 3 = 30$$

$$2 \times 5 \times 5 = 50$$

Is each of those products divisible by 2? Is each of those products divisible by 5? Is each of those products divisible by 10? How do you know? By examining a base-ten numeral for a whole number, how can you tell whether or not it names a number that is divisible by 10?

<i>A whole number is divisible by</i>	<i>if the ones digit of its simplest numeral is</i>
2	0, 2, 4, 6 or 8
5	0 or 5
10	0

Oral Tell which of the numerals below name whole numbers that are divisible by 2. By 5. By 10.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	72	35	134	490
2.	85	66	123	381
3.	84	95	152	272
4.	23	90	800	163
5.	98	27	780	754

Answer the following questions.

6. What are the possible remainders when a whole number is divided by 2?

7. What will be the remainder if an even number is divided by 2?

8. What are the possible remainders when a whole number is divided by 5?

9. What will be the remainder if a multiple of 5 is divided by 5?

10. If a number is divisible by 2 and by 5, is it divisible by 10?

Written Copy the following numerals.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	17	50	103	1094
2.	28	65	202	2115
3.	56	74	567	7532

4.	27	70	553	7770
5.	35	88	495	5052
6.	49	95	280	9995

Then do the following:

a. Draw a line under each numeral that names a number that is divisible by 2.

b. Draw a circle around each numeral that names a number that is divisible by 5.

c. Draw a check mark above each numeral that names a number that is divisible by 10.

Tell why If some whole number n is divisible by a whole number d , then $n \div d$ names a natural number. Tell why this statement is false.

Can you do this? The following numerals name the whole numbers in ascending order.

$0_{\text{four}}, 1_{\text{four}}, 2_{\text{four}}, 3_{\text{four}}, 10_{\text{four}}, 11_{\text{four}}, 12_{\text{four}}, 13_{\text{four}}, 20_{\text{four}}, 21_{\text{four}}, 22_{\text{four}}, \dots$

Can you discover a rule for telling when a base-four numeral names an even number?

The following numerals name whole numbers in ascending order.

$0_{\text{five}}, 1_{\text{five}}, 2_{\text{five}}, 3_{\text{five}}, 4_{\text{five}}, 10_{\text{five}}, 11_{\text{five}}, 12_{\text{five}}, 13_{\text{five}}, 14_{\text{five}}, 20_{\text{five}}, \dots$

Can you discover a rule for telling when a base-five numeral names an even number?

Divisibility by 3 and by 9

423

This number is divisible by 3.

$$4+2+3=9$$

This sum is divisible by 3.

5724

This number is divisible by 9.

$$5+7+2+4=18$$

This sum is divisible by 9.

In each example above, let us call the sum *the sum of the digits*.

A number is divisible by 3 if the sum of its digits is divisible by 3.

A number is divisible by 9 if the sum of its digits is divisible by 9.

Is 3246 divisible by 3?

Add: $3+2+4+6=15$ and $1+5=6$

Since 6 is divisible by 3,
3246 is divisible by 3.

Is 7485 divisible by 9?

Add: $7+4+8+5=24$ and $2+4=6$

Since 6 is not divisible by 9,
7485 is not divisible by 9.

Oral Tell how you would decide if 4362 is divisible by 3. By 9.

Written Copy the following numerals. Draw a line under each numeral that names a number divisible by 3. Draw a ring around each numeral that names a number divisible by 9.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	321	8275	2103	21072
2.	568	2142	5646	56349
3.	555	9918	8675	81256

Can you do this? Answer the following questions.

1. What is the least number that is divisible by 2 and by 3?
2. What is the least number that is divisible by 3 and by 5?
3. What is the least number that is divisible by 2, by 3, and by 5?

Tell why If a number is divisible by 9, then it is divisible by 3. If a number is divisible by 2 and by 3, then it is divisible by 6.

Prime Numbers

The following ways of naming the numbers show the factors of each number.

<u>2</u> 1×2	<u>3</u> 1×3	<u>4</u> 1×4 2×2	<u>5</u> 1×5	<u>6</u> 1×6 2×3	<u>7</u> 1×7
<u>8</u> 1×8 2×4 $2 \times 2 \times 2$	<u>9</u> 1×9 3×3	<u>10</u> 1×10 2×5	<u>11</u> 1×11	<u>12</u> 1×12 2×6 3×4 $2 \times 2 \times 3$	<u>13</u> 1×13

Which of the numbers named above have only themselves and 1 as factors? Such numbers are called **prime numbers**.

Any whole number greater than 1 that has only itself and 1 as factors is called a *prime number*, or simply a *prime*.

Which numbers named above are prime numbers?

Any whole number greater than 1 that is not a prime number is called a **composite number**.

Which numbers named above are composite numbers?

Oral Answer the questions below.

1. Is any multiple of 2, except 2 itself, a prime? Why or why not?
2. Is any multiple of 3, except 3 itself, a prime? Why or why not?
3. Is any multiple of 5, except 5 itself, a prime? Why or why not?

Written List the whole numbers from 2 through 100.

1. Circle 2 and cross out all other multiples of 2.
2. Do the same for 3, 5, and 7.
3. Now list all the prime numbers less than 100.

Prime Factors

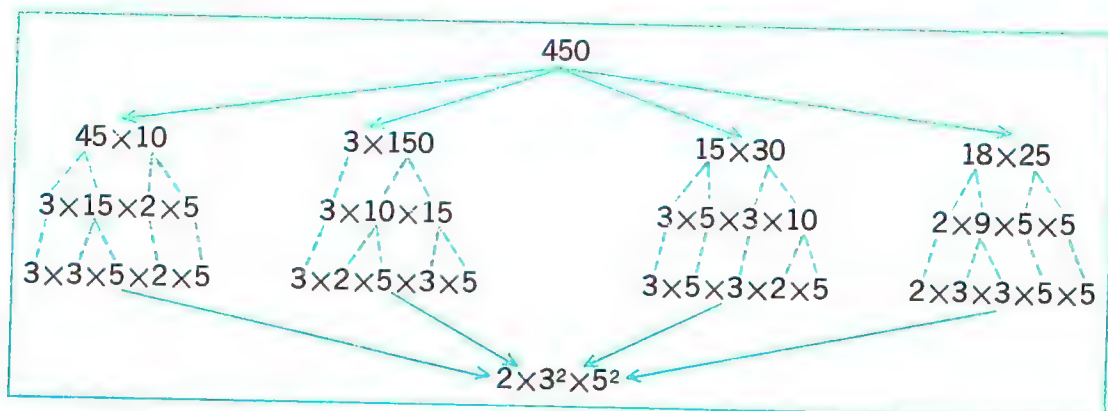
Since 3 is a prime number and also a factor of 15, you can say that 3 is a *prime factor* of 15. What is another prime factor of 15?

Every composite number can be expressed as a product of prime factors. You can find the prime factors of 450 in either of the ways shown below.

		450	
Prime Factors	2	225	450 is divisible by 2, so divide by 2.
	3	75	225 is not divisible by 2, but it is by 3.
	3	25	75 is not divisible by 2, but it is by 3.
	5	5	25 is not divisible by 2 or by 3, but it is by 5.
	5	1	5 is divisible by 5, so divide by 5.
	$2 \times 3 \times 3 \times 5 \times 5$		Therefore, $450 = 2 \times 3 \times 3 \times 5 \times 5$ or $2 \times 3^2 \times 5^2$.

How can you tell when to stop dividing in the above method?

You might recognize composite factors of 450. Then you can find the prime factors of these composite factors as shown below.



Are the prime factors the same regardless of which composite factors were chosen first?

When listing the prime factors of a number, let us agree to list them in order from the least to the greatest.

Oral Tell which number is named by each of the following expressions.

<i>a</i>	<i>b</i>	<i>c</i>
1. 3^2	2×5^2	$2^2 \times 3 \times 5$
2. 2^3	$2^3 \times 3$	$2 \times 3 \times 5^2$
3. 5^2	$3^2 \times 5$	$2^3 \times 5^2$

Answer questions 4–7 about the following example for finding the prime factors of 45.

$$\begin{array}{r|l} & 45 \\ 3 & 15 \\ 3 & 5 \\ 5 & 1 \end{array}$$

4. Why is 3 used as the first divisor instead of 2?

5. Why is 3 used again as the second divisor?

6. Why is the division stopped after dividing by 5?

7. Express 45 as a product of prime factors.

Tell how you would arrange the following prime factors from least to greatest. Use exponents whenever a factor is repeated.

<i>a</i>	<i>b</i>
8. $3 \times 2 \times 5 \times 2$	$3 \times 7 \times 2 \times 3 \times 3$
9. $5 \times 3 \times 2 \times 3$	$5 \times 2 \times 5 \times 2 \times 5$

Written Copy. Express each number as a product of prime factors. Arrange the prime factors from least to greatest.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1. 64	98	150	180
2. 20	36	120	121
3. 77	18	300	256
4. 42	40	288	315
5. 81	72	250	400

Can you do this? Each of the even numbers listed below is expressed as a sum of two prime numbers in at least one way.

$$8 = 5 + 3$$

$$16 = 11 + 5 \text{ or } 13 + 3$$

$$100 = 47 + 53 \text{ or } 97 + 3 \text{ or } 83 + 17 \text{ or } 71 + 29$$

Express each of the following even numbers as a sum of two primes.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1. 10	22	54	102
2. 18	26	66	124

In 1742 a mathematician named Goldbach made a guess that every even number, except 2, can be expressed as a sum of two primes. His guess has never been proved nor disproved. Check this for the even numbers less than 50.

Practice with Operations on Whole Numbers

Part 1 Name the property that is used in each step of solving the open sentences below.

$$\begin{array}{l} 1. \text{ ————— } n = (57 + 25) + 43 \\ 2. \text{ ————— } = (25 + 57) + 43 \\ \phantom{2. \text{ ————— }} = 25 + (57 + 43) \\ \phantom{2. \text{ ————— }} = 25 + 100 \text{ or } 125 \end{array}$$

$$\begin{array}{l} 3. \text{ ————— } k = 25(7 \times 4) \\ 4. \text{ ————— } = 25(4 \times 7) \\ \phantom{4. \text{ ————— }} = (25 \times 4)7 \\ \phantom{4. \text{ ————— }} = 100 \times 7 \text{ or } 700 \end{array}$$

Find the simplest numeral for each sum or product below.

a	b
5. $36 + 17 + 64$	$5 \times 19 \times 20$
6. $32 + 12 + 88$	$25 \times 9 \times 8$
7. $15 + 43 + 85$	$17 \times 40 \times 50$
8. $58 + 42 + 76$	$250 \times 13 \times 8$

Part 2 Write a reason for each step in proving that $2 \times 3 = 3 \times 2$.

$$\begin{array}{l} 2 \times 3 = 3 + 3 \\ 1. \text{ ————— } = (2 + 1) + (2 + 1) \\ 2. \text{ ————— } = (2 + 2) + (1 + 1) \\ 3. \text{ ————— } = 2 + 2 + 2 \\ 4. \text{ ————— } = 3 \times 2 \end{array}$$

Write a proof for each of the following statements.

a	b
5. $2 \times 7 = 7 \times 2$	$(3 \times 4)5 = 3(4 \times 5)$

Part 3 Write an open sentence for each problem below. Solve the open sentence. Answer the problem.

1. When in orbit, a satellite travels at a speed of 17,000 miles per hour. How many miles will it travel in 1 day?

2. John earned 82 cents and spent 17 cents. Now he needs 24 cents more to buy a record. How much does the record sell for?

3. The OK Garment Co. received an order for 384 white shirts. If the shirts are to be packed 12 per box, how many boxes will be needed to fill this order?

4. Mr. Bowden bought a new car for \$3435. He was allowed \$1450 for his old car, paid \$945 in cash, and signed a note for the rest. What was the amount of the note?

5. Fifteen boys were asked to deliver 3375 advertising bills. If each boy delivered the same number of bills, how many bills did each boy deliver?

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

Let a , b , and c represent whole numbers.

1. $a+0=a=0+a$ (52)
2. $a+b=b+a$ (52)
3. $(a+b)+c=a+(b+c)$ (53)
4. If $a+b=c$, then $a=c-b$ and $b=c-a$. (54)
5. $ab=ba$ (58)
6. $(ab)c=a(bc)$ (59)
7. $a \times 1=a=1 \times a$ (60)
8. $a(b+c)=ab+ac$ (62)
9. If $ab=c$, but $a \neq 0$ and $c \neq 0$, then $a=\frac{c}{b}$ and $b=\frac{c}{a}$. (66)
10. If a is greater than 1 and has only itself and 1 as factors, then a represents a prime number. (81)

Words to Know

1. Identity number of addition, commutative property of addition (52)
2. Associative property of addition (53)
3. Commutative property of multiplication (58)
4. Associative property of multiplication (59)

5. Identity number of multiplication (60)

6. Distributive property of multiplication over addition (62)

7. Prime number, composite number (81)

Questions to Discuss

1. How is the set of natural numbers different from the set of whole numbers? (51)
2. What two subtraction sentences can you write for $k+7=21$? (54)
3. What two division sentences can you write for $3n=18$? (66)
4. What does it mean to say that 21 is divisible by 7? (69)

Written Practice

Copy. Perform the operations.

a	b	c	
1. $\begin{array}{r} 564 \\ -328 \\ \hline \end{array}$	$\begin{array}{r} 6348 \\ -3277 \\ \hline \end{array}$	$\begin{array}{r} 70541 \\ -8375 \\ \hline \end{array}$	(57)

2. $\begin{array}{r} 257 \\ \times 6 \\ \hline \end{array}$	$\begin{array}{r} 3054 \\ \times 38 \\ \hline \end{array}$	$\begin{array}{r} 2538 \\ \times 1205 \\ \hline \end{array}$	(65)
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3. $3 \overline{)141}$	$16 \overline{)754}$	$254 \overline{)3328}$	(77)
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4. $2^3 \times 3$	$2^2 \times 5^3$	$2 \times 3^2 \times 5$	(83)
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Self-Evaluation

Part 1 Name the property that is illustrated by each of the following sentences.

- | a | b |
|----------------------|-------------------------|
| 1. $8 \times 1 = 8$ | $(6n)5 = 6(n \times 5)$ |
| 2. $5a = a \times 5$ | $3(a+b) = 3a+3b$ |
| 3. $0+3=3$ | $(3+1)+n=3+(1+n)$ |
| 4. $1+7=7+1$ | $5s+5t=5(s+t)$ |

Part 2 Copy. Find each sum or difference.

- | | a | b | c |
|----|--|--|--|
| 1. | $\begin{array}{r} 96 \\ +57 \\ \hline \end{array}$ | $\begin{array}{r} 524 \\ +672 \\ \hline \end{array}$ | $\begin{array}{r} 8324 \\ +7692 \\ \hline \end{array}$ |
| 2. | $\begin{array}{r} 367 \\ -42 \\ \hline \end{array}$ | $\begin{array}{r} 846 \\ -227 \\ \hline \end{array}$ | $\begin{array}{r} 5001 \\ -3051 \\ \hline \end{array}$ |
| 3. | $\begin{array}{r} 922 \\ -709 \\ \hline \end{array}$ | $\begin{array}{r} 779 \\ -584 \\ \hline \end{array}$ | $\begin{array}{r} 4223 \\ -2719 \\ \hline \end{array}$ |

Part 3 Copy. Find each product.

- | | a | b | c |
|----|---|--|--|
| 1. | $\begin{array}{r} 227 \\ \times 5 \\ \hline \end{array}$ | $\begin{array}{r} 381 \\ \times 30 \\ \hline \end{array}$ | $\begin{array}{r} 6245 \\ \times 26 \\ \hline \end{array}$ |
| 2. | $\begin{array}{r} 765 \\ \times 29 \\ \hline \end{array}$ | $\begin{array}{r} 785 \\ \times 402 \\ \hline \end{array}$ | $\begin{array}{r} 1267 \\ \times 529 \\ \hline \end{array}$ |
| 3. | $\begin{array}{r} 694 \\ \times 68 \\ \hline \end{array}$ | $\begin{array}{r} 873 \\ \times 420 \\ \hline \end{array}$ | $\begin{array}{r} 7967 \\ \times 3458 \\ \hline \end{array}$ |

Part 4 Copy. Find each quotient and remainder.

- | a | b | c |
|-----------------------|-----------------------|-------------------------|
| 1. $4 \overline{)76}$ | $34 \overline{)899}$ | $213 \overline{)16075}$ |
| 2. $5 \overline{)88}$ | $87 \overline{)984}$ | $553 \overline{)98217}$ |
| 3. $6 \overline{)92}$ | $31 \overline{)6574}$ | $607 \overline{)54038}$ |
| 4. $7 \overline{)81}$ | $26 \overline{)6727}$ | $428 \overline{)75000}$ |

Part 5 Copy. Express each of the following numbers as a product of prime factors. Use exponents and list the factors from least to greatest.

- | a | b | c |
|-------|-----|------|
| 1. 24 | 120 | 2000 |
| 2. 27 | 125 | 1500 |
| 3. 38 | 225 | 1000 |
| 4. 45 | 500 | 4000 |

Part 6 Copy. Solve each equation.

- | a | b |
|----------------------|---------------------------|
| 1. $26n=26$ | $t=(19+7)-7$ |
| 2. $k \div 5=0$ | $(19-12)+a=19$ |
| 3. $7x=112$ | $(12 \times 9) \div c=12$ |
| 4. $r \times 17=136$ | $(31-7) \div k=8$ |

Chapter 4

THE INTEGERS

Names for Numbers

W

Set of whole numbers

F

Set of fractional numbers

Your work in mathematics thus far has been concerned with the two sets of numbers indicated above.

Some of the numerals below name whole numbers. Which ones? Some of the numerals name fractional numbers. Which ones? One of the numerals names neither a whole number nor a fractional number. Which one?

$2+3$	$3+2$	$3-2$	$4\div 2$
2×3	$2-3$	10×3	
3×2	$2\div 4$	$10\div 3$	

A numeral names a number, and $2-3$ names neither a whole number nor a fractional number. Hence we need a more extended number system which includes numbers like $2-3$.

Oral Answer the questions below.

1. Do you think the number named by $2-3$ is more like the numbers in *W* or those in *F*?

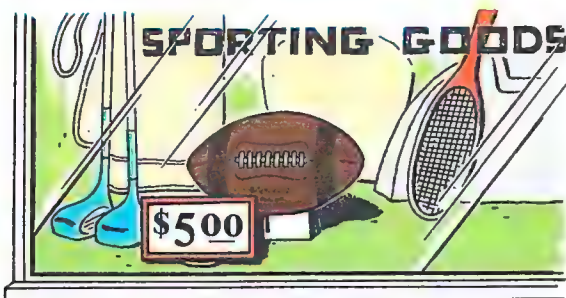
2. Which of the two sets of numbers, *W* or *F*, would you suggest we extend?

3. Which operation on the whole numbers 2 and 3 resulted in the number named by $2-3$?

Written For each numeral below, write *W* if it names a whole number, *F* if it names a fractional number, and *N* if it names neither of these.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$5+2$	$7\div 3$	$7-8$	$10\div 4$
2.	$9-9$	$3\div 7$	$6\div 2$	$5-8$
3.	$8-13$	$8+13$	$8\div 13$	8×13
4.	$12\div 4$	$4\div 12$	$4-12$	$4+12$

The Need for New Numbers



Jim has saved \$3.00 to buy the football shown at the left. The owner of the store knows Jim and trusts him to pay the rest of the money as soon as he can earn it. How would you explain Jim's financial situation at the moment he leaves the store?

To write an open sentence for this problem, you might think of subtracting the number of dollars Jim spent from the number of dollars he had. An open sentence for the problem might be

$$3 - 5 = n.$$

What does n stand for in this open sentence? Could you just as well have used k or r as the variable?

Does $3 - 5$ name a whole number? In a later lesson we shall create single numerals for such numbers.

Oral Discuss each problem situation given below.

1. On Tuesday noon the temperature was 7° above zero. By evening it had fallen 10° . What was the evening temperature?

2. The water level of a lake is normal when it is at the 0-mark. During the spring the water level rose 5 feet. But during the hot summer it fell 7 feet. How would you describe the water level then?

3. The floors of an apartment house are labeled B, 0, 1, 2, 3, 4.

The ground level is labeled 0 and basement level is labeled B. If you were to use a numeral instead of a letter, what numeral would you use instead of B?

4. Tom was watching the blastoff of a rocket on television. In the countdown he heard "T minus 4 seconds" and then the sound in the set went off for 6 seconds. What was the count when the sound came on again?

Written Write an open sentence for each problem in *Oral*.

A Number Line for Whole Numbers



You have used number lines to illustrate operations on the whole numbers. You may have also used number lines to illustrate properties of these operations.

What closed addition sentence is illustrated on the number line above?

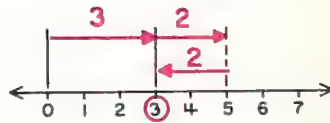
Use each number-line drawing below to solve the open sentence below it.



$$7 - 3 = a$$

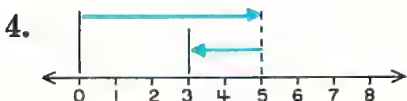
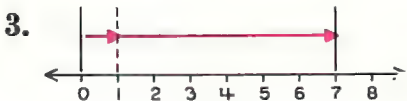
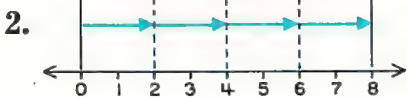
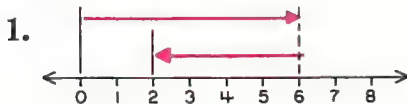


$$3 \times 2 = n$$



$$(3 + 2) - 2 = r$$

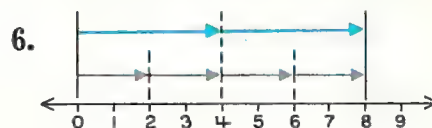
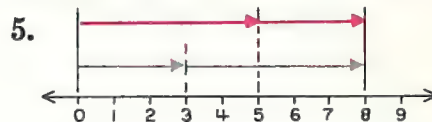
Oral Tell which operation on whole numbers is illustrated on each number line below.



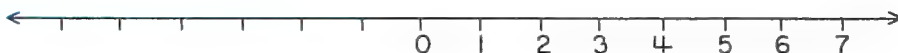
Written Do the following.

1-4. Write a closed sentence for each number line in *Oral* 1-4.

Write two closed sentences for each number line below. Then state which property of which operation is illustrated on that number line.



Extending a Number Line



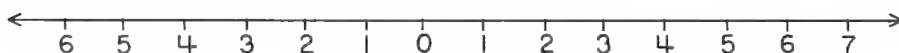
Since a straight line has no endpoints, it seems reasonable to extend the familiar number line to the left of the 0-mark.

It has been found convenient to make all segments on both sides of the 0-mark the same length. This means that if you move from 0 to 3 *to the right* you will be just as far from 0 as if you move from 0 to 3 *to the left*. Of course, the moves will be in *opposite directions*.

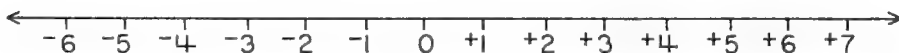
Recall that in problem situations that involved such numbers as 3–5 or 7–10, you also encountered “opposite directions.”

<i>above zero, below zero</i>	<i>before blastoff, after blastoff</i>
<i>to the north, to the south</i>	<i>above sea level, below sea level</i>

It seems reasonable that the points marked to the left of the 0-mark on the number line above could be associated with numbers like 3–5 or 7–10. But what numerals can we use to label these points? We might try the following.



If you were told to move from 0 to 4, would you know which way to move—to the left or to the right? Must you also know which direction to move? Then we might try the following.



Of the many symbols that have been suggested for telling both *distance* and *direction*, those shown on the last number line above have been generally accepted. The important thing to learn now is how to read and write these new numerals.

Notice that the small + and - symbols are used with the numerals for whole numbers to form these new numerals. These small symbols do *not* indicate operations, but rather they indicate *direction*.

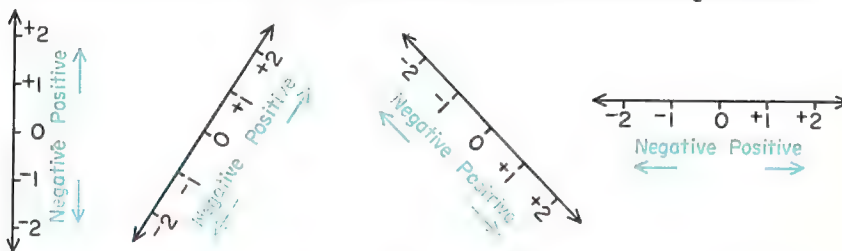
+3 is read *positive three*.

-4 is read *negative four*.

$2-3=-1$ is read *two minus three is equal to negative one*.

$+3+(-3)=0$ is read *positive three plus negative three is equal to zero*.

A number line need not be drawn in a horizontal position.



On which of these number lines might you think of a move to the right or to the left? On which of these number lines might you think of a move in the positive or in the negative direction?

Oral Read each of the following numerals. Then tell which point it refers to on a number line.

- | | <i>a</i> | <i>b</i> | <i>c</i> | <i>d</i> |
|----|----------|----------|----------|----------|
| 1. | +4 | -2 | +16 | -5 |
| 2. | +38 | -26 | +126 | -157 |

Read each sentence below.

- | | <i>a</i> | <i>b</i> |
|----|-----------|-----------|
| 3. | $+5+2=+7$ | $-6+4=-2$ |
| 4. | $-8+8=0$ | $+3-2=+5$ |

Written Refer to the last number line on page 90. Write a numeral to describe each of the following moves on the number line.

- | | <i>a</i> | <i>b</i> |
|----|---------------|---------------|
| 1. | from 0 to -2 | from 0 to +6 |
| 2. | from +3 to +6 | from -5 to -1 |
| 3. | from +2 to -1 | from +3 to 0 |
| 4. | from +4 to -4 | from -2 to -6 |
| 5. | from -5 to 0 | from +2 to +1 |

The Integers

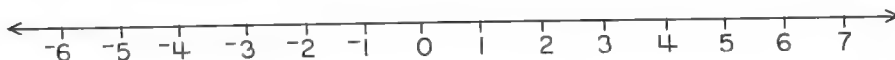
By extending the set of whole numbers, we create the new enlarged set of numbers named below.

The Set of **Integers** = $\{ \dots, -4, -3, -2, -1, 0, +1, +2, +3, +4, \dots \}$

Notice that the set of integers consists of three distinct sets of numbers.

- a. The set of *positive integers* $\{+1, +2, +3, \dots\}$
- b. The set of *negative integers* $\{-1, -2, -3, \dots\}$
- c. The set whose only member is 0 $\{0\}$

At the outset we used small symbols $+$ and $-$ in $+3$ and -3 to distinguish direction. Of course, you can also distinguish between 3 and -3 . If no confusion is possible, let us agree to use either $+3$ or 3 to name *positive three*. A number line could then be drawn as follows.



Which is farther to the right on the number line: 4 or 6 ? Which is greater: 4 or 6 ? Try this for other pairs of positive integers. You can express the order of the positive integers by either of the following.

$$1 < 2 < 3 < 4 < 5 < \dots \text{ or } \dots > 5 > 4 > 3 > 2 > 1$$

In order that this pattern holds for all integers, the following statement will be accepted.

The greater of two integers is represented farther to the right on a horizontal number line.

Since -3 is farther to the right than -5 on the number line above, you can say that $-3 > -5$ or that $-5 < -3$.

Since 0 is farther to the right than -2 on the number line above, you can say that $0 > -2$ or that $-2 < 0$.

Oral Answer the following questions.

1. How do you know that any positive integer is greater than zero? Greater than any negative integer?

2. How do you know that zero is greater than any negative integer?

3. How could you use a number line to tell which of two integers is the lesser?

For each pair of integers below, tell which is the greater. Refer to the number line on page 92 if necessary.

	<i>a</i>	<i>b</i>	<i>c</i>
4.	-4, -2	4, 2	-4, 2
5.	0, -3	3, 0	3, -3
6.	5, -1	-5, 1	-5, -1

Written Copy. Replace each ● with $<$, $=$, or $>$ so that the resulting sentence is true.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	7 ● 3	-3 ● -2	0 ● 4
2.	-1 ● -5	0 ● -3	17 ● 32
3.	-6 ● 1	9 ● 9	-9 ● 9
4.	8 ● -8	-8 ● 8	-8 ● -8
5.	1 ● -7	-3 ● -11	25 ● -37

Write a numeral to tell where on a number line you would be after the following moves.

6. from 0 to 3 and then 5 segments in the negative direction

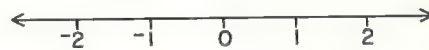
7. from 0 to 6 and then 6 segments in the negative direction

8. from 0 to -6 and then 4 segments in the positive direction

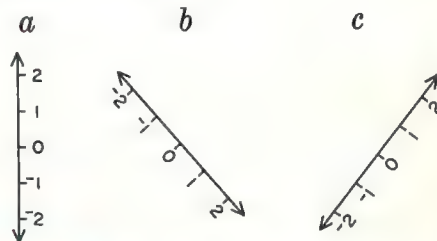
9. from 0 to -5 and then 5 segments in the positive direction

10. from 0 to -2 and then 3 segments in the negative direction

Can you do this? When a number line like the following is used, the greater number is represented farther to the right.



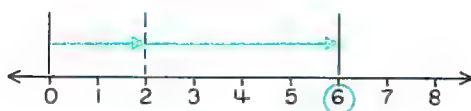
Make a similar statement for locating the greater number on each number line below.



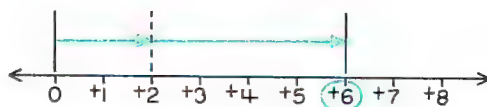
Tell why Why do we use only the symbol 0 for zero and not the symbols +0 and -0?

Addition of Integers

A



B



What addition of whole numbers is represented on the number line in A? Do both colored arrows point in the *same* direction? You can think of moving from 0 to 2 and then continuing in the *same* direction 4 more segments. You can express this as

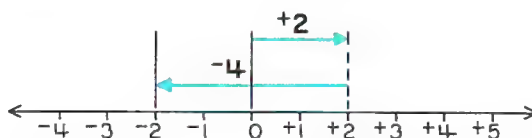
$$2+4=6.$$

The number-line drawing in B is almost identical to that in A. How are they different? In B, you can think of moving 2 segments in the positive direction from 0. Then move 4 more segments in the positive direction. You can express this as

$$+2+ +4= +6.$$

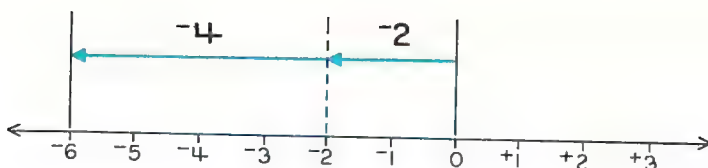
Since we have already agreed that numerals like 4 and +4 name the same number, the two equality sentences above make the same statement. Also, the two number lines above show the same addition.

In order to interpret addition of integers on a number line, study the following examples.



$$+2+ -4= -2$$

Start at the 0-mark. The first addend is +2, so move 2 segments in the positive direction. Since addition is indicated, continue by making the move indicated by the second addend. The second addend is -4, so move 4 segments in the negative direction. What is the sum of +2 and -4?

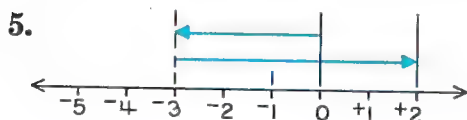
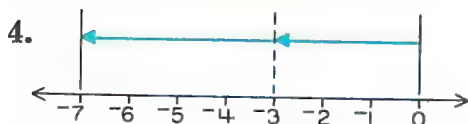
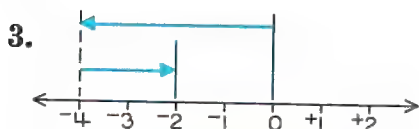
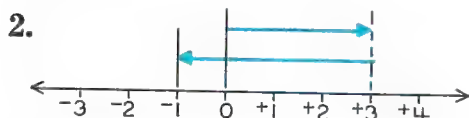
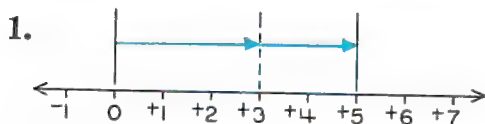


$$-2 + -4 = -6$$

Start at the 0-mark. The first addend is -2 , so move 2 segments in the negative direction. Since addition is indicated, continue by making the move indicated by the second addend. The second addend is -4 , so move 4 more segments in the negative direction. What is the sum of -2 and -4 ?

Addition on a number line can be interpreted as follows: Make the move indicated by the first addend, and then continue by making the move indicated by the second addend.

Oral Tell an addition sentence for each number line drawing below.



Tell how you could use a number line to determine the integer named by each of the following expressions.

	<i>a</i>	<i>b</i>	<i>c</i>
6.	$+1 + +3$	$+1 + -3$	$-1 + -3$
7.	$+3 + +1$	$-3 + +1$	$-3 + -1$

Written Do the following.

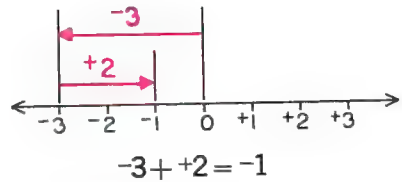
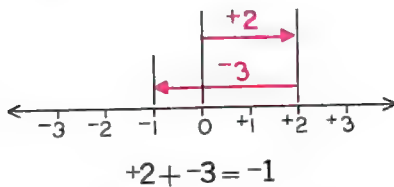
1–2. Write a single numeral for each sum in *Oral* 6–7. Use a number line if necessary.

Solve each open sentence below.

	<i>a</i>	<i>b</i>
3.	$-3 + -6 = a$	$+7 + -2 = b$
4.	$+4 + +9 = e$	$-14 + +8 = n$
5.	$-13 + -17 = x$	$-13 + +17 = y$
6.	$+20 + -30 = m$	$-8 + +8 = k$

Properties of Addition

When a number system is extended, it is desirable that the properties of the original system remain valid in the new system. Let us investigate some properties of addition.



Are the same two integers used as addends in both addition examples above? Is the order of the addends changed? What property of addition appears to be true for the integers?

$$(-3 + +2) + +4 = -1 + +4 = +3$$

$$-3 + (+2 + +4) = -3 + +6 = +3$$

Are the same three integers used in the same order as addends in these two addition examples? Are the addends associated in different ways? What property of addition appears to be true for the integers?

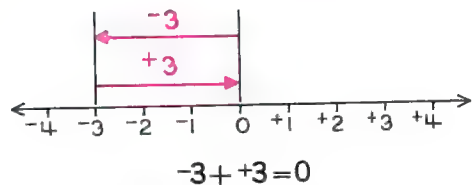
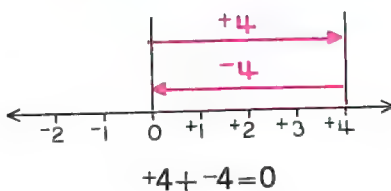
If a , b , and c represent integers, then

$$a + b = b + a$$

Commutative Property of Addition

$$(a + b) + c = a + (b + c).$$

Associative Property of Addition



Integers like $+4$ and -4 are called **opposites**. That is, $+4$ is the *opposite* of -4 and -4 is the *opposite* of $+4$. What appears to be the sum of any integer and its opposite?

If a represents a positive integer, then

$$a + -a = 0 \text{ and } -a + a = 0.$$

Sum of Opposite Integers

Oral Tell what integer is named by each of the following expressions.

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|-----------|-----------|-----------|
| 1. | $-3 + -4$ | $+7 + -5$ | $+7 + +5$ |
| 2. | $-4 + -3$ | $-5 + +7$ | $+5 + +7$ |
| 3. | $+4 + -3$ | $-7 + +5$ | $-7 + -5$ |
| 4. | $-3 + +4$ | $+5 + -7$ | $-5 + -7$ |

Tell which property of addition of integers is illustrated by each sentence below.

5. $(+3 + -2) + -7 = +3 + (-2 + -7)$
6. $-17 + +17 = 0$
7. $+8 + -6 = -6 + +8$
8. $-15 + -9 = -9 + -15$
9. $+39 + -39 = 0$
10. $-6 + (-4 + -3) = (-6 + -4) + -3$

Written Solve each open sentence below. If necessary, use a number line drawing to help you solve the equation.

- | | <i>a</i> | <i>b</i> |
|----|---------------|---------------|
| 1. | $-3 + -6 = a$ | $-6 + -3 = b$ |
| 2. | $+5 + -4 = n$ | $-4 + +5 = k$ |
| 3. | $-8 + +2 = r$ | $+2 + -8 = t$ |

4. $-7 + +7 = y$ $+7 + -7 = m$
5. $+17 + -3 = x$ $-3 + +17 = u$

Without using a number line, solve each open sentence below. Choose the easiest way to solve each open sentence.

6. $-2 + (+4 + -6) = a$
7. $(+3 + -5) + +5 = b$
8. $-7 + (-3 + -4) = x$
9. $(+2 + +3) + -4 = y$
10. $-15 + (-7 + +8) = t$
11. $(-6 + +2) + -2 = r$

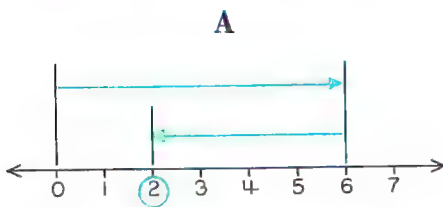
Can you do this? Tell whether zero is still the identity number of addition for integers. (*Hint:* Think of a zero addend on the number line as "stay where you are.")

Tell why Find the sum for each row, for each column, and along each diagonal in the square below.

+12	-16	+4
-8	0	+8
-4	+16	-12

Why is this array called a *Magic Square*?

Subtraction of Integers



What subtraction of whole numbers is represented on the number line in **A**? You can think of moving from 0 to 6, and then moving 4 segments in the *opposite* direction. You can express this as

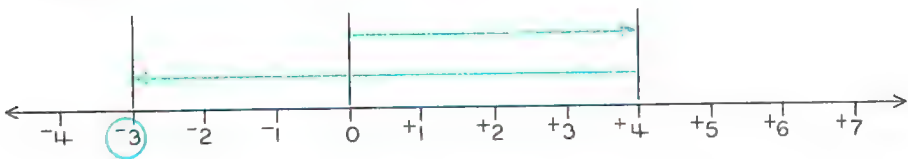
$$6 - 4 = 2.$$

In **B**, you can think of moving 6 segments in the positive direction from 0. Then move 4 segments in the opposite or negative direction. You can express this as

$$+6 - +4 = +2.$$

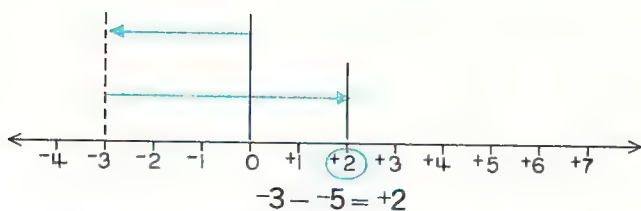
The subtrahend in $+6 - +4 = +2$ is $+4$, which indicates a move of 4 segments in the positive direction. Since subtraction is the inverse of addition, you should expect to make that move in the *opposite direction* from that indicated by $+4$.

In order to interpret subtraction of integers on a number line, study the following examples.



$$+4 - +7 = -3$$

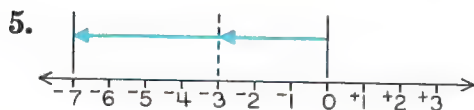
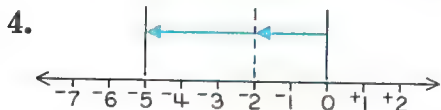
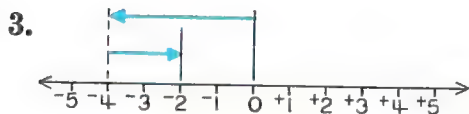
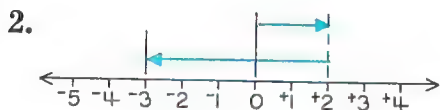
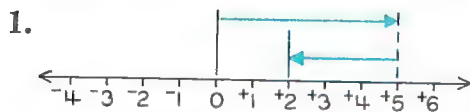
Start at the 0-mark. The minuend is $+4$, so move 4 segments in the positive direction. Since subtraction is indicated, continue by making the move indicated by the subtrahend, *but in the opposite direction* from that indicated by $+7$. The subtrahend is $+7$, so move 7 segments in the negative direction. What integer is named by $+4 - +7$?



Start at the 0-mark. The minuend is -3 , so move 3 segments in the negative direction. Since subtraction is indicated, continue by moving as many segments as indicated by the subtrahend, but *in the opposite direction*. The subtrahend is -5 , so move 5 segments in the positive direction.

Subtraction on a number line can be interpreted as follows: Make the move indicated by the minuend, and then continue by making the move indicated by the subtrahend, but *in the opposite direction* from that indicated by the subtrahend.

Oral Tell a subtraction sentence for each number-line drawing below.



Tell how you could use a number line to determine the integer named by each expression below.

a	b	c
6. $+3 - +5$	$+3 - -5$	$-3 - +5$

7. $-7 - +3$	$+7 - +3$	$+7 - -3$
--------------	-----------	-----------

Written Do the following.

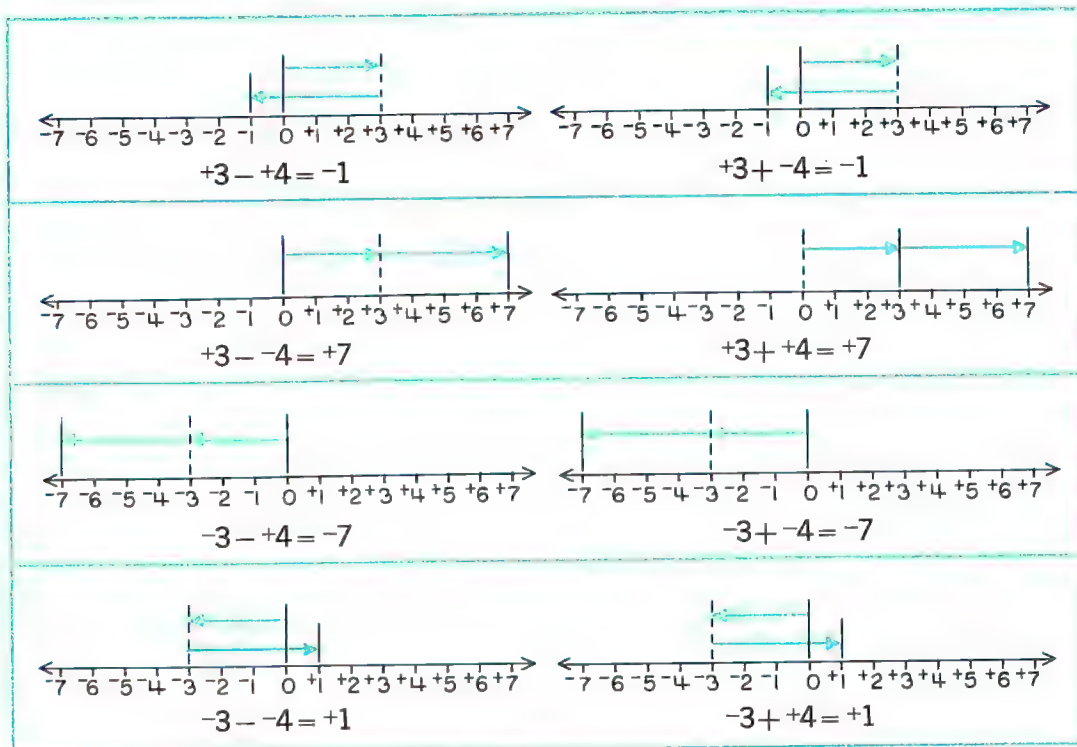
1-2. Write a single numeral for each sum in *Oral* 6-7.

Solve each open sentence below.

a	b
3. $-8 - +3 = x$	$+2 - +7 = y$
4. $+17 - -5 = n$	$-12 - +4 = m$
5. $-9 - -3 = a$	$+8 - +8 = r$

Subtraction of Integers

Is subtraction of integers necessary? The number-line drawings below will help you answer this question.



Are the first two number lines identical? Are the first two sentences identical? If not, how are they different?

Answer the same two questions for each of the other pairs of number lines and closed sentences above to discover how subtraction of integers can be changed to addition of integers.

Let $+a$ and $-a$ represent any two opposite integers.

Subtracting $+a$ has the same result as adding $-a$.

Subtracting $-a$ has the same result as adding $+a$.

$+3 - -4 = n$ can be changed to $+3 + +4 = n$.

$-3 - +4 = k$ can be changed to $-3 + -4 = k$.

Oral Tell what numeral should replace the blank so that each of the following sentences becomes true.

a

b

1. $+5 - -3 = +5 + \underline{\quad}$ $+5 - +3 = +5 + \underline{\quad}$

2. $-7 - +2 = -7 + \underline{\quad}$ $-7 - -2 = -7 + \underline{\quad}$

3. $+9 - +4 = +9 + \underline{\quad}$ $-1 - -3 = -1 + \underline{\quad}$

4. $-6 - -1 = -6 + \underline{\quad}$ $+8 - +7 = +8 + \underline{\quad}$

Tell how you would change each subtraction sentence below to an addition sentence.

a

b

5. $+5 - +2 = n$ $+5 - -2 = m$

6. $r = -7 - +8$ $t = -7 - -8$

7. $x - +2 = +5$ $y - -4 = -7$

Written Write an open addition sentence for each subtraction sentence below. Solve the open addition sentence.

a

b

1. $+11 - +3 = a$ $-9 - -6 = c$

2. $+3 - +11 = x$ $-6 - -9 = d$

3. $-8 - +17 = n$ $+5 - -24 = s$

4. $+17 - -8 = r$ $-24 - +5 = t$

5. $+29 - +29 = k$ $-13 - -13 = w$

Write an open sentence for each problem. Solve the open sentence. Answer the problem.

6. A submarine submerged and went down 550 feet. Then it rose 210 feet. Describe its position now.

7. At noon the temperature was 11° . During the afternoon it rose 7° , and during the night it fell 23° . What was the temperature then?

8. A football team got the ball on their own 48-yard line. In the next two plays they gained 3 yards and lost 7 yards. On which yard line were they then?

Another way Study the following example. Give a reason for each step.

$$\begin{aligned} +51 - +37 &= +51 + -37 \\ &= (+11 + +40) + -37 \\ &= +11 + (+40 + -37) \\ &= +11 + +3 \text{ or } +14 \end{aligned}$$

Use this method to solve each open sentence below.

a

b

1. $+20 - +7 = a$ $-17 + -5 = c$

2. $-32 - -12 = x$ $+47 - +15 = y$

3. $+31 - +17 = n$ $-29 + -17 = r$

Tell why If x stands for some integer, you don't know whether $-x$ stands for a negative integer or for a positive integer. Why not?

Multiplication of Integers

Let us assume the following properties for integers.

If a , b , and c represent any integers, then

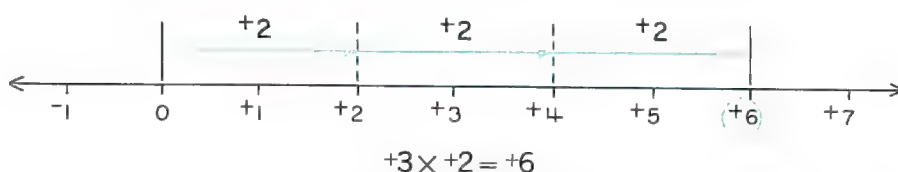
$$ab = ba$$

Commutative Property of Multiplication

$$a(b+c) = ab+ac.$$

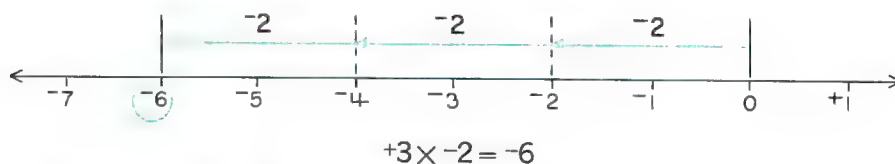
Distributive Property of Multiplication over Addition

Now let us develop a definition for multiplication of integers. First, suppose both factors are positive integers.



You can interpret the number-line drawing above as $+2 + +2 + +2 = +6$ or as $3 \times +2 = +6$. According to our agreement about 3 and $+3$, you can replace $3 \times +2 = +6$ with $+3 \times +2 = +6$. Is multiplication of positive integers like multiplication of whole numbers?

Next, suppose one factor is a positive integer and the other factor is a negative integer.



You can interpret this number-line drawing as $3 \times -2 = -6$ or as $+3 \times -2 = -6$. Since we assume that multiplication of integers is commutative, $+3 \times -2 = -2 \times +3$. Hence, $-2 \times +3 = -6$.

Study the following examples to discover a pattern for finding the product of a positive integer and a negative integer.

$$-4 \times +5 = -20 \quad +6 \times -7 = -42 \quad +9 \times -8 = -72 \quad -3 \times +6 = -18$$

Is each product a negative integer? How does multiplication of whole numbers help you find each product?

Now, suppose both factors are negative integers. To find the product of -3 and -2 , let us begin as follows.

$-3 \times 0 = 0$	If a factor is 0, the product is 0.
$-3 \times (+2 + -2) = 0$	Rename 0 as a sum of opposites.
$(-3 \times +2) + (-3 \times -2) = 0$	Distributive Property
$-6 + (-3 \times -2) = 0$	Multiplication of Integers

The sum of -6 and (-3×-2) is 0, so -6 and (-3×-2) must be *opposite integers*. Then what integer does -3×-2 name?

Therefore, $-3 \times -2 = +6$.

We have discovered the following about multiplication of integers.

- If both factors are positive, the product is positive.
- If both factors are negative, the product is positive.
- If one factor is positive and the other is negative, the product is negative.

Oral Tell how you might use the table below in finding the product of two integers.

1.

\times	$+$	$-$
$+$	$+$	$-$
$-$	$-$	$+$

Tell which integer is named by each expression below.

- | | a | b | c |
|----|----------------|----------------|----------------|
| 2. | $+3 \times +4$ | $+3 \times -4$ | $-3 \times +4$ |
| 3. | -7×-6 | $-7 \times +6$ | -9×-3 |

Written Copy. Find the simplest numeral for each product.

- | | a | b | c |
|----|----------------|-----------------|------------------|
| 1. | $-5 \times +6$ | $+7 \times +4$ | -3×-6 |
| 2. | $+6 \times -5$ | $+4 \times +7$ | -6×-3 |
| 3. | $+9 \times -4$ | -8×-2 | -7×-2 |
| 4. | $-4 \times +9$ | -2×-8 | -2×-7 |
| 5. | -12×0 | $+17 \times -3$ | $-5 \times +22$ |
| 6. | -6×-8 | $-1 \times +7$ | -1×-7 |
| 7. | $+9 \times -5$ | -7×-4 | $-12 \times +10$ |

Division of Integers

Multiplication and division of whole numbers are inverse operations. How is this shown in the following examples?

$$3 \times 5 = 15 \text{ so } 15 \div 5 = 3$$

$$7 \times 4 = 28 \text{ so } 28 \div 7 = 4$$

Multiplication and division of integers are to be inverse operations also. This is shown in the examples below.

$$+3 \times +2 = +6 \text{ so } +6 \div +2 = +3$$

$$+3 \times -2 = -6 \text{ so } -6 \div -2 = +3$$

\div	+	-
+	+	-
-	-	+

If the dividend and the divisor are both positive or both negative, the quotient is positive.

$$-3 \times +2 = -6 \text{ so } -6 \div +2 = -3$$

$$-3 \times -2 = +6 \text{ so } +6 \div -2 = -3$$

\div	+	-
+	+	-
-	-	+

If either the dividend or the divisor is positive and the other is negative, the quotient is negative.

Oral Tell how you would find the simplest numeral for each expression below. Tell whether that numeral would have a + or a - sign.

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|---------------|---------------|---------------|
| 1. | $+8 \div +2$ | $-8 \div +2$ | $+8 \div -2$ |
| 2. | $-6 \div +3$ | $+6 \div -3$ | $-6 \div -3$ |
| 3. | $-10 \div +5$ | $-10 \div -5$ | $+10 \div -5$ |
| 4. | $+21 \div -7$ | $+21 \div +7$ | $-21 \div -7$ |

Written Do the following.

1-4. Express each quotient in *Oral* 1-4 as a single integer.

Can you do this? Find the simplest numeral for each expression below.

- | | <i>a</i> | <i>b</i> |
|----|---------------------------|-------------------------|
| 1. | $(+4 \times -3) \div -6$ | $+21 - (+8 \div -4)$ |
| 2. | $(+3 - +9) \div -2$ | $(-12 \div -4) + -7$ |
| 3. | $-28 \div (+2 \times -7)$ | $-16 \div (-8 \div -2)$ |

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Numerals like 2-5 or 7-10 and many physical problems exhibit a need for extending the set of whole numbers. (87-88)

2. A number line can be extended to show numbers that are neither whole nor fractional numbers. (90-91)

3. The name for an integer, like -5, indicates both a direction and a distance. (90)

4. The greater of two integers is represented farther to the right on a horizontal number line. (92)

5. Addition of integers is commutative and associative. (96)

6. The sum of opposite integers is zero. (96)

7. Subtracting an integer has the same result as adding its opposite. (100)

Words to Know

1. Positive, negative (91)
2. Set of integers, positive integers, negative integers (92)
3. Opposite integers (96)

Questions to Discuss

1. How would you find the sum of +2 and -5? (94)

2. How would you find the sum of -2 and -5? (95)

3. Subtraction of integers is not necessary. Why not? (100)

4. How would you solve the open sentence $+5 - -7 = n$? (100)

5. How would you find the product of +4 and -6? Of -4 and -6? (102-103)

6. How would you solve the open sentence $-12 \div +4 = r$? (104)

Written Practice

Solve each open sentence.

- | | <i>a</i> | <i>b</i> |
|----|--------------------|--------------------------|
| 1. | $-4 + -3 = a$ | $-8 + +5 = x$ (95) |
| 2. | $-5 + +5 = c$ | $+7 + -7 = d$ (96) |
| 3. | $+2 - +9 = n$ | $-8 - -4 = k$ (99) |
| 4. | $+5 - -3 = k$ | $-7 - +2 = n$ (99) |
| 5. | $+5 \times -6 = s$ | $-4 \times -7 = t$ (102) |
| 6. | $-12 \div +6 = y$ | $-18 \div -2 = e$ (104) |

Self-Evaluation

Part 1 Draw a number line for the integers from -10 to $+10$. Use this number line to write a numeral for each of the following moves.


- | a | b |
|----------------------|-------------------|
| 1. from 0 to -3 | from 0 to $+5$ |
| 2. from -2 to -7 | from $+3$ to $+7$ |
| 3. from -1 to $+4$ | from $+6$ to -9 |

Use the number line you drew to tell the resulting position for each of the following combinations of moves. Start at 0 for each combination.

4. 3 segments in the positive direction followed by 6 segments in the negative direction

5. 4 segments in the positive direction followed by 3 segments in the positive direction

6. 2 segments in the negative direction followed by 10 segments in the positive direction

Part 2 Copy. Replace each  by $<$, $=$, or $>$ so the resulting sentence is true.

- | a | b | c |
|------------------------------|----------------------------|---------------------------|
| 1. $+3 \text{ } \text{ } +7$ | $+3 \text{ } \text{ } -7$ | $+7 \text{ } \text{ } -7$ |
| 2. $-5 \text{ } \text{ } -5$ | $-2 \text{ } \text{ } 0$ | $+1 \text{ } \text{ } -4$ |
| 3. $0 \text{ } \text{ } +9$ | $+3 \text{ } \text{ } -16$ | $-7 \text{ } \text{ } +1$ |

Part 3 Tell which property of the operations on integers is illustrated by each sentence below.

- $+2 + (-3 + -5) = (+2 + -3) + -5$
- $-17 + +32 = +32 + -17$
- $-31 + +31 = 0$
- $(+3 \times -5) \times -2 = +3 \times (-5 \times -2)$
- $-4(+5 + -7) = (-4 \times +5) + (-4 \times -7)$

Part 4 Copy. Find the simplest numeral for each sum.

- | a | b |
|--------------|------------------|
| 1. $+7 + -3$ | $-3 + (-2 + +5)$ |
| 2. $-5 + -6$ | $+7 + (+4 + -6)$ |

Copy. Find the simplest numeral for each difference.

- | a | b | c |
|--------------|-----------|------------|
| 3. $+9 - +3$ | $-8 - +5$ | $+7 - -8$ |
| 4. $+2 - +7$ | $-7 - -7$ | $-6 - -10$ |

Copy. Find the simplest numeral for each product or quotient.

- | a | b | c |
|-------------------|----------------|----------------|
| 5. $-3 \times +4$ | $+5 \times -2$ | -5×-2 |
| 6. -8×-3 | $+4 \times +7$ | $+9 \times -1$ |
| 7. $+16 \div +8$ | $-16 \div +8$ | $-16 \div -8$ |

Chapter 5

THE RATIONAL NUMBERS

The Need for Extending the Number System

$$15 \times 3 = a$$

$$8 - 3 = n$$

$$+6 \div +3 = p$$

$$+2 \div +3 = r$$

$$27 + 16 = b$$

$$7 - 9 = m$$

$$-10 \div +2 = q$$

$$-5 \div +2 = s$$

Are 15×3 and $27 + 16$ names for whole numbers? Are the product and the sum of two whole numbers always whole numbers?

Is $8 - 3$ a name for a whole number? Is $7 - 9$ a name for a whole number? Since the difference of two whole numbers is not always a whole number, it was necessary to extend the number system to include the set of integers.

Set of Integers = $\{ \dots, -4, -3, -2, -1, 0, +1, +2, +3, +4, \dots \}$

Is $+6 \div +3$ a name for an integer? Is $-10 \div +2$ a name for an integer? Are $+2 \div +3$ and $-5 \div +2$ names for integers? In order that sentences like $+2 \div +3 = r$ and $-5 \div +2 = s$ may have solutions, we must extend the number system beyond the set of integers.

Oral Which open sentences below have solutions in $\{0, 1, 2, 3, \dots\}$?

a

b

c

1. $7 + 8 = a$ $6 - 9 = b$ $5 \times 14 = c$

2. $15 - 7 = k$ $w + 16 = 5$ $5x = 15$

Which open sentences below have solutions in the set of integers?

a

b

3. $-7 + -7 = c$

$+16 - -4 = s$

4. $+6 \times -4 = n$

$-25 \div 6 = w$

Written Copy. Using the set of integers as the replacement set, find the solution set for each open sentence.

a

b

1. $-15 - -5 = k$

$+19 \times -5 = b$

2. $-27 \div +3 = f$

$+17 - +7 = a$

3. $(-23)(-4) = c$

$+14 + -7 = d$

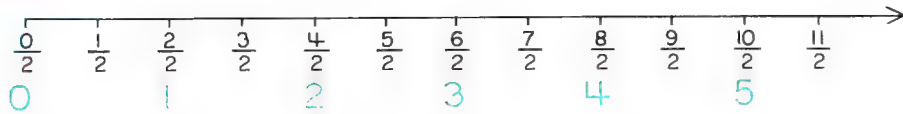
4. $-16 + +7 = e$

$+31 \div -7 = h$

5. $-19 \div +4 = h$

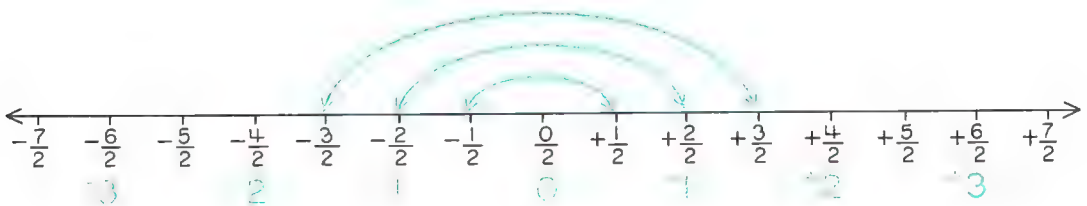
$(-5)(-3) = j$

Fractions



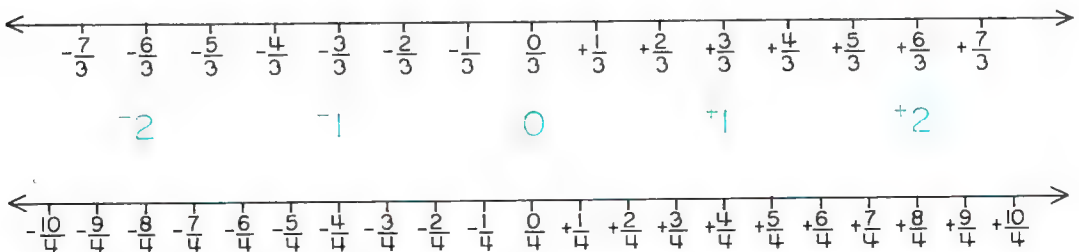
You are already familiar with representing fractional numbers on a number line as shown above. Recall that symbols like $\frac{0}{2}$, $\frac{1}{2}$, $\frac{2}{2}$, and so on are called *fractions* and are names for numbers.

The number line for whole numbers was extended to show both positive and negative integers. In the same manner, the number line above can be extended to show both positive and negative numbers named by fractions.



How would you read -1 and $+1$? Similarly, $-\frac{1}{2}$ and $+\frac{1}{2}$ are read *negative one half* and *positive one half*. Are both $+\frac{1}{2}$ and $-\frac{1}{2}$ the same distance from 0? How is the direction from 0 indicated in $+\frac{1}{2}$? In $-\frac{1}{2}$? For each positive number shown on the number line above, is there a negative number shown the same distance but in the opposite direction from 0?

You can also construct number lines for other numbers, as shown below for thirds and fourths.



Oral Tell how you would read each of the following fractions.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$-\frac{5}{2}$	$+\frac{3}{3}$	$-\frac{5}{4}$	$+\frac{3}{8}$
2.	$+\frac{4}{7}$	$-\frac{2}{6}$	$+\frac{1}{5}$	$-\frac{11}{9}$
3.	$-\frac{15}{3}$	$+\frac{7}{10}$	$-\frac{7}{7}$	$+\frac{13}{3}$

Tell *how far* and *in which direction* you would move from 0 to represent each of the following numbers on a number line.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
4.	$+\frac{3}{2}$	$\frac{0}{4}$	$-\frac{7}{16}$	$+\frac{11}{2}$
5.	$-\frac{4}{6}$	$+\frac{7}{9}$	$-\frac{12}{4}$	$-\frac{13}{7}$
6.	$+\frac{5}{8}$	$+\frac{7}{7}$	$-\frac{15}{6}$	$-\frac{9}{5}$

For each number named below, name another number that is represented the same distance from 0 on the number line.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
7.	$+\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{5}{6}$	$-\frac{9}{4}$
8.	$+\frac{3}{2}$	$+\frac{7}{8}$	$+\frac{15}{16}$	$+\frac{9}{10}$
9.	$-\frac{4}{5}$	$-\frac{2}{3}$	$-\frac{5}{7}$	$-\frac{9}{12}$

Answer the following questions.

10. If the number named by a fraction is one, what can you say about the numerator and the denominator named in the fraction?

11. Why is zero considered neither a positive nor a negative number?

Written Draw number lines and label those points which represent the following sets of numbers.

- $\{ \dots, -\frac{2}{5}, -\frac{1}{5}, \frac{0}{5}, +\frac{1}{5}, +\frac{2}{5}, \dots \}$
- $\{ \dots, -\frac{2}{6}, -\frac{1}{6}, \frac{0}{6}, +\frac{1}{6}, +\frac{2}{6}, \dots \}$
- $\{ \dots, -\frac{2}{7}, -\frac{1}{7}, \frac{0}{7}, +\frac{1}{7}, +\frac{2}{7}, \dots \}$
- $\{ \dots, -\frac{2}{8}, -\frac{1}{8}, \frac{0}{8}, +\frac{1}{8}, +\frac{2}{8}, \dots \}$

Refer to the number lines on page 108. Write a numeral to indicate the distance and the direction for each of the following moves on a number line.

	<i>a</i>	<i>b</i>
5.	from 0 to $+\frac{1}{2}$	from 0 to $-\frac{2}{3}$
6.	from $+\frac{1}{2}$ to $+\frac{4}{2}$	from $-\frac{1}{4}$ to $-\frac{3}{4}$
7.	from $-\frac{2}{3}$ to $+\frac{2}{3}$	from $+\frac{3}{4}$ to $-\frac{2}{4}$
8.	from $-\frac{5}{4}$ to $-\frac{4}{4}$	from $+\frac{5}{3}$ to $+\frac{2}{3}$
9.	from $-\frac{3}{4}$ to 0	from $+\frac{2}{3}$ to 0

Can you do this? Show the following by using number lines.

- Integers can be named by fractions.
- A number can be named by more than one fraction.

Rational Numbers

$$+8 \div +2 = +4$$

$$+8 \div -2 = -4$$

$$-8 \div +2 = -4$$

$$-8 \div -2 = +4$$

You already know how to find the quotient of two integers as shown above. Since the number *four* can be named in many different ways, including $8 \div 2$, the division sentences above can be stated as follows.

$$+8 \div +2 = +(8 \div 2)$$

$$+8 \div -2 = -(8 \div 2)$$

$$-8 \div +2 = -(8 \div 2)$$

$$-8 \div -2 = +(8 \div 2)$$

A fraction may be used to express division. For example, $8 \div 2$ can also be expressed as $\frac{8}{2}$. This idea can be used to express the division sentences above in still another way.

$$\frac{+8}{+2} = +\left(\frac{8}{2}\right)$$

$$\frac{+8}{-2} = -\left(\frac{8}{2}\right)$$

$$\frac{-8}{+2} = -\left(\frac{8}{2}\right)$$

$$\frac{-8}{-2} = +\left(\frac{8}{2}\right)$$

The parentheses are used in expressions like $-(\frac{8}{2})$ to indicate *negative* $\frac{8}{2}$. We can avoid the use of () if we agree that *negative* $\frac{8}{2}$ can be named by $-\frac{8}{2}$. Notice that the symbol $-$ is aligned with the fraction line and not with one of the numerals in the fraction. The same idea applies in expressing $+(\frac{8}{2})$ as *positive* $\frac{8}{2}$ or as $+\frac{8}{2}$.

You can now express the quotient of any two integers as indicated in the following examples.

$$-5 \div +3 = \frac{-5}{+3} = -\left(\frac{5}{3}\right) = -\frac{5}{3}$$

$$-7 \div -2 = \frac{-7}{-2} = +\left(\frac{7}{2}\right) = +\frac{7}{2}$$

$-\frac{5}{3}$ is read *positive five over negative three*.

Numerals like those below name **rational numbers**.

$$\frac{+2}{+7}$$

$$\frac{+8}{-2}$$

$$\frac{-5}{+3}$$

$$\frac{-7}{-2}$$

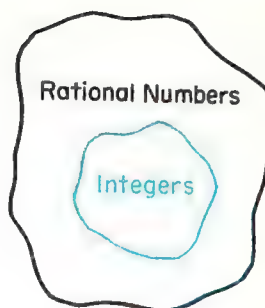
If p and q represent integers and q does not represent zero, then $\frac{p}{q}$ represents a *rational number*.

The term *rational* can be associated with the term *ratio* since fractions can be used to express ratios. Hence, any rational number can be named by a fraction.

Since the integer $+4$ can be named

$\frac{+4}{+1}$, $\frac{+8}{+2}$, $\frac{-4}{-1}$, $\frac{-8}{-2}$, and so on,

the integer $+4$ is considered to be a rational number. Are all integers also rational numbers? The diagram at the right shows that the set of integers is a subset of the set of rational numbers.



Just as we agreed that 3 and $+3$ name the same integer, let us agree that $\frac{3}{4}$ and $+\frac{3}{4}$ name the same rational number. In the rest of this chapter we will name positive rational numbers by numerals like $\frac{7}{8}$ instead of numerals like $+\frac{7}{8}$.

Oral Tell how you would read each of the following fractions.

- | | a | b | c | d |
|----|-----------------|-----------------|------------------|------------------|
| 1. | $\frac{+3}{+2}$ | $\frac{-3}{+2}$ | $\frac{-3}{-2}$ | $\frac{3}{2}$ |
| 2. | $\frac{+3}{-2}$ | $-\frac{7}{8}$ | $\frac{-5}{+12}$ | $\frac{+9}{-10}$ |

Written Do the following.

1-4. Write a single fraction like $\frac{4}{5}$ or $-\frac{4}{5}$ that names the same rational number as each division numeral in *Oral* 4-7.

Copy. Find the solution set for each open sentence. Use the set of rational numbers as the replacement set.

Tell a fraction that names the same number as each division numeral below.

- | | a | b | c |
|----|----------------|----------------|----------------|
| 3. | $+16 \div +5$ | $-20 \div +37$ | $+27 \div -10$ |
| 4. | $+15 \div -7$ | $-41 \div -7$ | $+13 \div +4$ |
| 5. | $-18 \div +11$ | $+7 \div -24$ | $-15 \div -7$ |
| 6. | $-37 \div -14$ | $+14 \div +3$ | $-13 \div +2$ |
| 7. | $+17 \div -8$ | $-3 \div +19$ | $-21 \div -5$ |

- | | a | b |
|-----|--------------------|-------------------|
| 5. | $+16 \div +12 = n$ | $+13 \div +4 = n$ |
| 6. | $+9 \div +2 = n$ | $+9 \div -2 = n$ |
| 7. | $-13 \div +4 = n$ | $-13 \div -4 = n$ |
| 8. | $-7 \div +8 = n$ | $-18 \div -3 = n$ |
| 9. | $-21 \div +7 = n$ | $+19 \div -5 = n$ |
| 10. | $+32 \div -8 = n$ | $-32 \div -8 = n$ |

Multiplication of Rational Numbers

$$\text{A } 2 \times 3 = 6$$

$$\text{B } \frac{6}{3} \times \frac{12}{4} = 6$$

Which number is the first factor in **A**? Is the first factor in **B** the same number? What is the second factor in **A**? Is the second factor in **B** the same number? Why should the product be the same in both cases?

You can find the product of $\frac{6}{3}$ and $\frac{12}{4}$ as shown below.

$$\frac{6}{3} \times \frac{12}{4} = \frac{6 \times 12}{3 \times 4} = \frac{72}{12} = 6$$

How are 72 and 12 obtained? How is 6 obtained from $\frac{72}{12}$?

The example above illustrates the following method of finding the product of two rational numbers.

If $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers, then

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \quad \text{or} \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

You can think of the product of two rational numbers like $\frac{3}{4}$ and $\frac{7}{8}$ as shown below.

Product of the Numerators

$$\frac{3}{4} \times \frac{7}{8} = \frac{\cancel{3} \times 7}{4 \times 8} = \frac{21}{32}$$

Product of the Denominators

Are such numbers as 7, +5, and -9 integers? Are they also rational numbers? Since integers are also rational numbers, they can be renamed as fractions when finding products.

$$7 \times \frac{3}{4} = \frac{7}{1} \times \frac{3}{4} = \frac{7 \times 3}{1 \times 4} = \frac{21}{4}$$

$$\frac{1}{3} \times 5 = \frac{1}{3} \times \frac{5}{1} = \frac{1 \times 5}{3 \times 1} = \frac{5}{3}$$

Why was $\frac{7}{1}$ chosen as a name for 7 instead of $\frac{14}{2}$, $\frac{21}{3}$ or $\frac{28}{4}$? Why is $\frac{5}{1}$ a convenient name to use for the factor 5 above?

Oral Explain each example below.

a	b
1. $n = \frac{1}{5} \times \frac{2}{3}$ $= \frac{1 \times 2}{5 \times 3}$ $= \frac{2}{15}$	$n = \frac{3}{7} \times \frac{4}{5}$ $= \frac{3 \times 4}{7 \times 5}$ $= \frac{12}{35}$
2. $n = 3 \times \frac{3}{4}$ $= \frac{3}{1} \times \frac{3}{4}$ $= \frac{3 \times 3}{1 \times 4}$ $= \frac{9}{4}$	$n = \frac{2}{7} \times 6$ $= \frac{2}{7} \times \frac{6}{1}$ $= \frac{2 \times 6}{7 \times 1}$ $= \frac{12}{7}$

Tell how you would rename each whole number factor below. Give a reason for your choice. Then tell how you would find each product as a single fraction.

a	b	c
3. $\frac{3}{4} \times 5$	$2 \times \frac{3}{5}$	$\frac{2}{3} \times 7$
4. $3 \times \frac{3}{5}$	$\frac{1}{2} \times 3$	$7 \times \frac{5}{6}$

Tell how you would name each product as a single fraction. Then tell how you decided upon your answer.

a	b	c
5. $\frac{3}{4} \times \frac{5}{2}$	$\frac{5}{8} \times \frac{1}{2}$	$\frac{1}{4} \times \frac{9}{8}$
6. $\frac{7}{4} \times \frac{5}{2}$	$\frac{2}{3} \times \frac{5}{7}$	$\frac{3}{4} \times \frac{3}{2}$
7. $\frac{3}{5} \times \frac{7}{9}$	$\frac{5}{8} \times \frac{1}{6}$	$\frac{7}{8} \times \frac{5}{4}$
8. $\frac{1}{5} \times \frac{7}{12}$	$\frac{1}{3} \times \frac{1}{15}$	$\frac{4}{7} \times \frac{8}{9}$
9. $\frac{2}{9} \times \frac{8}{3}$	$\frac{5}{11} \times \frac{6}{7}$	$\frac{9}{2} \times \frac{7}{16}$
10. $\frac{3}{5} \times \frac{4}{9}$	$\frac{7}{8} \times \frac{5}{6}$	$\frac{9}{10} \times \frac{3}{7}$

Written Follow these directions.

1-6. Express each product in *Oral* 5-10 as a single fraction.

Copy the following. Then solve each equation by using the set of rational numbers as the replacement set.

a	b	c
7. $\frac{3}{8} \times \frac{9}{10} = n$	$\frac{5}{6} \times \frac{7}{8} = n$	$\frac{13}{7} \times \frac{5}{6} = n$
8. $\frac{17}{20} \times \frac{3}{8} = n$	$\frac{4}{9} \times \frac{1}{21} = n$	$\frac{8}{9} \times \frac{12}{13} = n$
9. $\frac{3}{2} \times \frac{25}{53} = n$	$\frac{4}{3} \times \frac{20}{31} = n$	$\frac{9}{5} \times \frac{12}{17} = n$
10. $\frac{20}{30} \times \frac{17}{19} = n$	$\frac{16}{7} \times \frac{12}{5} = n$	$\frac{8}{17} \times \frac{9}{11} = n$
11. $\frac{17}{23} \times \frac{4}{5} = n$	$\frac{12}{7} \times \frac{9}{5} = n$	$\frac{13}{6} \times \frac{7}{15} = n$
12. $\frac{11}{4} \times \frac{3}{29} = n$	$\frac{3}{13} \times \frac{8}{19} = n$	$\frac{2}{3} \times \frac{19}{21} = n$
13. $\frac{3}{7} \times \frac{2}{5} = n$	$\frac{5}{8} \times \frac{3}{2} = n$	$\frac{5}{11} \times \frac{4}{9} = n$
14. $\frac{3}{4} \times \frac{5}{12} = n$	$\frac{1}{7} \times \frac{1}{12} = n$	$\frac{8}{7} \times 3 = n$

Tell how How would you express the number named by each product below as a single fraction?

a	b
1. $(+\frac{1}{5}) \times (+\frac{3}{4})$	$(+\frac{3}{4}) \times (-\frac{1}{2})$
2. $(-\frac{2}{7}) \times (+\frac{2}{3})$	$(-\frac{3}{4}) \times (-\frac{1}{2})$
3. $(+\frac{1}{3}) \times (-5)$	$-7 \times (+\frac{3}{4})$
4. $(-\frac{5}{3}) \times (-\frac{2}{3})$	$(-\frac{7}{8}) \times (+5)$

Properties of Multiplication of Rational Numbers

A $6 \times 2 = 2 \times 6$

B $+6 \times -2 = -2 \times +6$

What property of multiplication is illustrated for whole numbers in **A**? For integers in **B**?

$$\frac{5}{6} \times \frac{1}{2} = \frac{5 \times 1}{6 \times 2} \\ = \frac{5}{12}$$

$$\frac{1}{2} \times \frac{5}{6} = \frac{1 \times 5}{2 \times 6} \\ = \frac{5}{12}$$

Does commuting two rational number factors change the product? Do you think multiplication of rational numbers is commutative?

If $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers, then

$$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}.$$

C $(2 \times 3) \times 5 = 2 \times (3 \times 5)$

D $(+2 \times -4) \times +6 = +2 \times (-4 \times +6)$

What property of multiplication is illustrated for whole numbers in **C**? For integers in **D**?

$$\left(\frac{1}{3} \times \frac{2}{5}\right) \times \frac{4}{3} = \frac{1 \times 2}{3 \times 5} \times \frac{4}{3} \\ = \frac{2}{15} \times \frac{4}{3} \\ = \frac{2 \times 4}{15 \times 3} \\ = \frac{8}{45}$$

$$\frac{1}{3} \times \left(\frac{2}{5} \times \frac{4}{3}\right) = \frac{1}{3} \times \frac{2 \times 4}{5 \times 3} \\ = \frac{1}{3} \times \frac{8}{15} \\ = \frac{1 \times 8}{3 \times 15} \\ = \frac{8}{45}$$

Does associating rational number factors in different ways change the product? Do you think multiplication of rational numbers is associative?

If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ represent rational numbers, then

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right).$$

$$\begin{aligned}\frac{5}{6} \times 1 &= \frac{5}{6} \times \frac{1}{1} \\ &= \frac{5 \times 1}{6 \times 1} \\ &= \frac{5}{6}\end{aligned}$$

$$\begin{aligned}1 \times \frac{6}{7} &= \frac{1}{1} \times \frac{6}{7} \\ &= \frac{1 \times 6}{1 \times 7} \\ &= \frac{6}{7}\end{aligned}$$

In extending the number system to the set of rational numbers, the number one is retained as the identity number of multiplication. How is this shown in the above examples?

If $\frac{a}{b}$ represents a rational number, then

$$1 \times \frac{a}{b} = \frac{a}{b} = \frac{a}{b} \times 1.$$

Oral State the expression that results when the factors are commuted in each product named below.

a	b	c
1. $\frac{3}{4} \times \frac{7}{8}$	$\frac{4}{5} \times \frac{1}{3}$	$\frac{2}{3} \times \frac{5}{7}$
2. $\frac{3}{8} \times \frac{3}{4}$	$\frac{5}{6} \times \frac{7}{8}$	$\frac{1}{7} \times \frac{3}{5}$
3. $\frac{7}{3} \times \frac{4}{9}$	$\frac{3}{7} \times \frac{2}{5}$	$\frac{9}{10} \times \frac{3}{7}$
4. $\frac{4}{5} \times \frac{6}{7}$	$\frac{1}{2} \times \frac{3}{7}$	$\frac{8}{9} \times \frac{4}{5}$

Tell how you could associate the factors in two different ways in each product named below.

a	b	c
5. $\frac{3}{5} \times \frac{1}{2} \times \frac{7}{8}$	$\frac{1}{3} \times \frac{2}{5} \times \frac{4}{3}$	$\frac{2}{3} \times \frac{5}{7} \times \frac{8}{9}$
6. $\frac{4}{3} \times \frac{1}{7} \times \frac{2}{5}$	$\frac{1}{2} \times \frac{5}{7} \times \frac{3}{4}$	$\frac{4}{5} \times \frac{2}{3} \times \frac{1}{7}$
7. $\frac{5}{6} \times \frac{1}{3} \times \frac{7}{8}$	$\frac{2}{3} \times \frac{2}{5} \times \frac{4}{7}$	$\frac{7}{8} \times \frac{3}{5} \times \frac{1}{6}$
8. $\frac{8}{9} \times \frac{4}{5} \times \frac{2}{3}$	$\frac{1}{6} \times \frac{5}{8} \times \frac{7}{10}$	$\frac{8}{9} \times \frac{1}{7} \times \frac{2}{5}$

Written Do the following.

1-4. Copy *Oral* 1-4. Express each product as a single fraction. Then commute the factors and again express the product as a single fraction.

5-8. Copy each exercise in *Oral* 5-8 twice. Place () to associate the factors in two different ways. Then express each product as a single fraction.

Tell why Why are each of the following numerals names for the rational number zero?

$$\frac{0}{-5}, \frac{0}{+7}, \frac{0}{+3}$$

Do any of the following fractions name a number? Why or why not? Can 0 ever name a denominator?

$$\frac{-5}{0}, \frac{+7}{0}, \frac{+13}{0}$$

Greatest Common Factor

You can express 36 and 60 as products in order to find their factor sets.

$$36 = \begin{array}{l} 1 \times 36 \\ 2 \times 18 \\ 3 \times 12 \\ 4 \times 9 \\ 6 \times 6 \end{array}$$

Factor set of 36

$\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$$60 = \begin{array}{l} 1 \times 60 \\ 2 \times 30 \\ 3 \times 20 \\ 4 \times 15 \\ 5 \times 12 \\ 6 \times 10 \end{array}$$

Factor set of 60

$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

Which factors do 36 and 60 have in common? Which of the common factors is the greatest? Therefore, we call 12 the **greatest common factor** of 36 and 60.

Another way of finding the greatest common factor of 36 and 60 is shown below. This method can be used to find the greatest common factor of two or more numbers.

Express the numbers as products of powers of prime factors.	$36 = 2 \times 2 \times 3 \times 3$ $= 2^2 \times 3^2$	$60 = 2 \times 2 \times 3 \times 5$ $= 2^2 \times 3^1 \times 5^1$
Choose the least power of each common prime factor.	2^2 and 3^1	
Find the product of these powers of prime factors.	$2^2 \times 3^1 = 4 \times 3 = 12$	
The greatest common factor of 36 and 60 is 12.		

What is the greatest common factor of 14 and 15?

$$14 = 2 \times 7 \text{ or } 1 \times 14$$

$$15 = 3 \times 5 \text{ or } 1 \times 15$$

$$\text{Factor set of } 14 = \{1, 2, 7, 14\}$$

$$\text{Factor set of } 15 = \{1, 3, 5, 15\}$$

Two numbers like 14 and 15 are said to be **relatively prime**.

If the greatest common factor of two numbers is 1, the two numbers are said to be *relatively prime*.

Oral Express each number below as a product of powers of prime factors.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	8	12	15	16
2.	21	24	26	28
3.	30	32	35	36

Name the least power of each common prime factor listed in each pair of expressions below.

	<i>a</i>	<i>b</i>
4.	$2^3 \times 3, 2^2 \times 3^2$	$2 \times 3, 2^2 \times 3 \times 7$
5.	$5 \times 7, 2^2 \times 5^2$	$2 \times 3, 2 \times 5 \times 7$
6.	$2^3 \times 3, 2 \times 3^2$	$2^2 \times 3, 2^3 \times 3 \times 5$

Tell how the greatest common factor is found in each example below.

7.	$\begin{array}{c} 30 \\ \swarrow \quad \searrow \\ 2 \times 3 \times 5 \end{array}$	$\begin{array}{c} 50 \\ \swarrow \quad \searrow \\ 2 \times 5^2 \end{array}$
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The greatest common factor of 30 and 50 is 2×5 or 10.

8.	$\begin{array}{c} 18 \\ \swarrow \quad \searrow \\ 2 \times 3^2 \end{array}$	$\begin{array}{c} 27 \\ \\ 3^3 \end{array}$	$\begin{array}{c} 36 \\ \swarrow \quad \searrow \\ 2^2 \times 3^2 \end{array}$
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The greatest common factor of 18, 27, and 36 is 3^2 or 9.

9.	$\begin{array}{c} 21 \\ \swarrow \quad \searrow \\ 3 \times 7 \end{array}$	$\begin{array}{c} 25 \\ \\ 5^2 \end{array}$
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The greatest common factor of 21 and 25 is 1.

Tell which numbers in each set below are relatively prime.

	<i>a</i>	<i>b</i>
10.	{8,9,12}	{2,5,8}
11.	{3,8,15}	{4,6,9}
12.	{7,4,12}	{5,7,10}

Written Find the greatest common factor of the numbers in each set below.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	{9,27}	{12,9}	{2,4,6}
2.	{16,8}	{8,27}	{5,10,25}
3.	{8,12}	{40,50}	{8,10,12}
4.	{16,24}	{15,22}	{9,12,24}
5.	{18,24}	{18,15}	{6,7,13}
6.	{20,25}	{30,48}	{6,8,12}
7.	{15,16}	{16,27}	{10,15,30}
8.	{42,60}	{11,13}	{8,16,20}

Can you do this? You can express -10 as a product of integers in any of the following ways.

$$-1 \times +10, -2 \times +5, +1 \times -10, +2 \times -5$$

List the factor set of -10. Then express -20 as a product of integers in all possible ways. Finally, find the greatest common factor of -10 and -20.

Simplest Form

What is the product of 1 and any other rational number? You can replace 1 by any convenient fraction to find many names for a rational number such as $\frac{2}{3}$ as shown below.

$$\frac{2}{3} \times 1 = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6} \quad \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{3}{3} = \frac{6}{9} \quad \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{8}{12} \quad \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{5}{5} = \frac{10}{15}$$

What is the greatest common factor of the numerator and the denominator in $\frac{10}{15}$? In $\frac{8}{12}$? In $\frac{6}{9}$? In $\frac{4}{6}$? In $\frac{2}{3}$? In which fraction are the numerator and denominator relatively prime? For this reason, we say $\frac{2}{3}$ is a fraction in **simplest form**.

A fraction is in simplest form if the numerator and the denominator are relatively prime.

You have seen how a fraction like $\frac{10}{15}$, which is not in simplest form, is obtained from the fraction $\frac{2}{3}$, which is in simplest form. Let's see how $\frac{10}{15}$ can be renamed in simplest form.

$\frac{10}{15} = \frac{2 \times 5}{3 \times 5}$	since	$\left\{ \begin{array}{l} 2 \times 5 = 10 \quad \longrightarrow \quad 10 \div 5 = 2 \\ \text{and} \quad \text{Inverse operation} \quad \text{and} \\ 3 \times 5 = 15 \quad \longrightarrow \quad 15 \div 5 = 3 \end{array} \right\}$	so	$\frac{10 \div 5}{15 \div 5} = \frac{2}{3}$
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Knowing that $\frac{2}{3} = \frac{10}{15}$ and that $\frac{2}{3} = \frac{10 \div 5}{15 \div 5}$ you can conclude by the transitive property of equality that $\frac{10}{15} = \frac{10 \div 5}{15 \div 5}$. The computation can be carried out as follows.

$$\frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$$

Notice that $\frac{2}{3}$ is obtained from $\frac{10}{15}$ by dividing both the numerator and the denominator named in $\frac{10}{15}$ by their greatest common factor. This idea can be stated as follows.

If a and b represent integers and k represents their greatest common factor, then

$$\frac{a}{b} = \frac{a \div k}{b \div k}$$

Oral Tell which fractions below are in simplest form. Tell which are not. Then tell how you decided.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$\frac{2}{4}$	$\frac{7}{8}$	$\frac{6}{9}$	$\frac{1}{2}$
2.	$\frac{5}{6}$	$\frac{8}{10}$	$\frac{5}{7}$	$\frac{12}{16}$
3.	$\frac{8}{12}$	$\frac{11}{24}$	$\frac{8}{14}$	$\frac{4}{5}$
4.	$\frac{1}{3}$	$\frac{5}{15}$	$\frac{5}{18}$	$\frac{14}{20}$

Tell how you could use the identity number of multiplication to obtain the second fraction in each pair from the first.

	<i>a</i>	<i>b</i>	<i>c</i>
5.	$\frac{1}{2}, \frac{6}{12}$	$\frac{3}{4}, \frac{15}{20}$	$\frac{2}{3}, \frac{16}{24}$
6.	$\frac{1}{6}, \frac{2}{12}$	$\frac{2}{5}, \frac{6}{15}$	$\frac{1}{2}, \frac{10}{20}$
7.	$\frac{7}{8}, \frac{14}{16}$	$\frac{1}{3}, \frac{7}{21}$	$\frac{1}{4}, \frac{4}{16}$

Explain how each fraction below is changed to simplest form.

<i>a</i>	<i>b</i>
8. $\frac{10}{16} = \frac{10 \div 2}{16 \div 2} = \frac{5}{8}$	$\frac{12}{24} = \frac{12 \div 12}{24 \div 12} = \frac{1}{2}$

By what number would you divide each numerator and denominator to express each fraction below in simplest form? Tell how you found this number.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
9.	$\frac{3}{6}$	$\frac{5}{15}$	$\frac{7}{21}$	$\frac{9}{12}$
10.	$\frac{4}{6}$	$\frac{2}{12}$	$\frac{10}{16}$	$\frac{15}{25}$

Written Copy. Change each fraction to simplest form.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$\frac{2}{4}$	$\frac{4}{6}$	$\frac{2}{6}$	$\frac{3}{9}$
2.	$\frac{2}{12}$	$\frac{2}{10}$	$\frac{2}{16}$	$\frac{14}{16}$
3.	$\frac{2}{8}$	$\frac{4}{10}$	$\frac{3}{12}$	$\frac{6}{16}$
4.	$\frac{8}{12}$	$\frac{4}{8}$	$\frac{5}{10}$	$\frac{6}{8}$
5.	$\frac{9}{18}$	$\frac{12}{16}$	$\frac{10}{15}$	$\frac{24}{32}$
6.	$\frac{8}{10}$	$\frac{4}{12}$	$\frac{6}{12}$	$\frac{6}{10}$
7.	$\frac{4}{16}$	$\frac{6}{18}$	$\frac{8}{16}$	$\frac{9}{12}$
8.	$\frac{10}{12}$	$\frac{10}{16}$	$\frac{18}{27}$	$\frac{8}{24}$
9.	$\frac{14}{24}$	$\frac{10}{8}$	$\frac{20}{24}$	$\frac{7}{7}$
10.	$\frac{16}{12}$	$\frac{11}{11}$	$\frac{24}{8}$	$\frac{16}{6}$
11.	$\frac{42}{14}$	$\frac{64}{16}$	$\frac{45}{60}$	$\frac{42}{48}$
12.	$\frac{75}{15}$	$\frac{34}{34}$	$\frac{16}{80}$	$\frac{45}{54}$

Another way You can express $\frac{10}{16}$ in simplest form as shown below.

$$\frac{10}{16} = \frac{2 \times 5}{2 \times 8} = \frac{2}{2} \times \frac{5}{8} = 1 \times \frac{5}{8} \text{ or } \frac{5}{8}$$

Why is $\frac{10}{16}$ changed to $\frac{2 \times 5}{2 \times 8}$ instead of $\frac{2 \times 5}{4 \times 4}$ or $\frac{1 \times 10}{8 \times 2}$? How is the greatest common factor used in changing $\frac{10}{16}$ to $\frac{2 \times 5}{2 \times 8}$? How is $\frac{5}{8}$ obtained from $\frac{2 \times 5}{2 \times 8}$?

Use the above method to express each number named in *Oral* 9–10 as a fraction in simplest form.

Products in Simplest Form

You could find the product of $\frac{7}{12}$ and $\frac{9}{14}$ in simplest form as shown below.

$$\frac{7}{12} \times \frac{9}{14} = \frac{7 \times 9}{12 \times 14} = \frac{63}{168} = \frac{3 \times 21}{8 \times 21} = \frac{3}{8} \times \frac{21}{21} = \frac{3}{8} \times 1 = \frac{3}{8}$$

When using the above method, it may not be easy to determine that 21 is the greatest common factor of the numerator and the denominator. Let us start over to see if we can simplify the computation.

$$\frac{7}{12} \times \frac{9}{14} = \frac{7 \times 9}{12 \times 14}$$

Notice that 7 is a factor of the numerator and that 14 is a factor of the denominator. What is the greatest common factor of 7 and 14? Hence, it seems reasonable to divide both the numerator and the denominator of $\frac{7 \times 9}{12 \times 14}$ by 7 at this step rather than later in the computation. In fact, this can be done with the original product $\frac{7}{12} \times \frac{9}{14}$. Does a similar situation exist for 9 and 12?

You can use these ideas to shorten the computation in finding the product of $\frac{7}{12}$ and $\frac{9}{14}$ in simplest form.

Divide the numerator and denominator by 7.	Divide the numerator and denominator by 3.	Find the product.
$\frac{\overset{1}{\cancel{7}}}{12} \times \frac{9}{\underset{2}{\cancel{14}}}$	$\frac{\overset{1}{\cancel{7}}}{12} \times \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{14}}}$	$\frac{\overset{1}{\cancel{7}}}{12} \times \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{14}}} = \frac{1 \times 3}{4 \times 2} = \frac{3}{8}$

This method can be used to find the product of more than two rational numbers.

Divide the numerator and denominator by 4.	Divide the numerator and denominator by 5.	Divide the numerator and denominator by 5.
$\frac{\overset{1}{\cancel{4}}}{5} \times \frac{\underset{2}{\cancel{5}}}{8} \times \frac{15}{12}$	$\frac{\overset{1}{\cancel{4}}}{5} \times \frac{\overset{1}{\cancel{5}}}{8} \times \frac{15}{12}$	$\frac{\overset{1}{\cancel{4}}}{5} \times \frac{\overset{1}{\cancel{5}}}{8} \times \frac{\overset{5}{\cancel{15}}}{12} = \frac{1 \times 1 \times 5}{1 \times 2 \times 4} = \frac{5}{8}$

Oral Explain how each product below is changed to simplest form.

a	b
1. $\overset{1}{\cancel{3}} \times \overset{5}{\underset{4}{\cancel{12}}} = \frac{5}{28}$	$\overset{1}{\cancel{2}} \times \overset{3}{\underset{8}{\cancel{16}}} = \frac{3}{8}$
2. $\overset{1}{\cancel{5}} \times \overset{3}{\underset{2}{\cancel{9}}} = \frac{3}{8}$	$\overset{2}{\cancel{14}} \times \overset{2}{\underset{3}{\cancel{21}}} = \frac{4}{9}$

Explain how you would change each product below to simplest form.

a	b	c
3. $\frac{6}{7} \times \frac{9}{10}$	$\frac{3}{4} \times \frac{16}{21}$	$\frac{9}{10} \times \frac{10}{9}$
4. $\frac{8}{15} \times \frac{25}{12}$	$\frac{15}{12} \times \frac{6}{10}$	$\frac{6}{3} \times \frac{9}{8}$
5. $\frac{24}{21} \times \frac{14}{12}$	$\frac{35}{27} \times \frac{9}{14}$	$\frac{15}{16} \times \frac{12}{21}$

Explain how you would change each product below to simplest form.

a	b
6. $\frac{4}{5} \times \frac{7}{2} \times \frac{15}{8}$	$\frac{8}{9} \times \frac{3}{10} \times \frac{5}{12}$
7. $\frac{3}{8} \times \frac{4}{9} \times \frac{6}{11}$	$\frac{7}{12} \times \frac{1}{14} \times \frac{18}{5}$
8. $\frac{6}{15} \times \frac{2}{3} \times \frac{5}{16}$	$\frac{2}{3} \times \frac{9}{10} \times \frac{5}{7}$

Written Copy. Change each product to simplest form.

a	b	c
1. $\frac{3}{4} \times \frac{4}{7}$	$\frac{7}{5} \times \frac{5}{7}$	$\frac{5}{8} \times \frac{3}{10}$
2. $\frac{6}{5} \times \frac{10}{12}$	$\frac{3}{8} \times \frac{12}{15}$	$\frac{7}{12} \times \frac{18}{28}$

3. $\frac{5}{9} \times 9$	$27 \times \frac{7}{9}$	$\frac{36}{40} \times \frac{5}{9}$
4. $\frac{9}{10} \times \frac{5}{24}$	$\frac{4}{5} \times \frac{5}{12}$	$\frac{42}{25} \times \frac{15}{14}$
5. $\frac{9}{11} \times \frac{33}{18}$	$\frac{5}{8} \times \frac{8}{5}$	$24 \times \frac{5}{8}$

Copy. Express each product in simplest form.

a	b
6. $\frac{5}{7} \times \frac{9}{10} \times \frac{21}{11}$	$\frac{3}{10} \times \frac{5}{8} \times \frac{7}{12}$
7. $\frac{5}{6} \times \frac{6}{7} \times \frac{7}{5}$	$\frac{12}{15} \times 10 \times \frac{9}{16}$
8. $\frac{9}{28} \times \frac{12}{13} \times \frac{21}{27}$	$\frac{3}{4} \times \frac{6}{7} \times \frac{14}{9}$

Can you do this? It is always possible to express an integer like -3 as -1×3 . This pattern also holds for all rational numbers. Therefore, $-\frac{3}{4} = -1 \times \frac{3}{4}$.

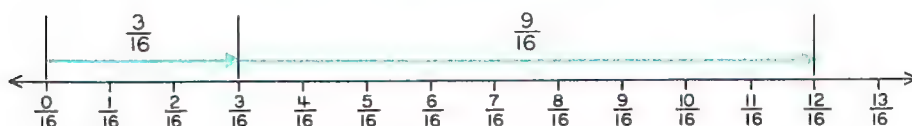
Study how it is shown below that $-\frac{3}{4} \times \frac{1}{2} = -\frac{3}{8}$.

$$\begin{aligned}
 -\frac{3}{4} \times \frac{1}{2} &= -1 \times \frac{3}{4} \times \frac{1}{2} \\
 &= -1 \times \left(\frac{3}{4} \times \frac{1}{2} \right) \\
 &= -1 \times \frac{3}{8} \text{ or } -\frac{3}{8}
 \end{aligned}$$

1. Show that $\frac{3}{7} \times (-\frac{5}{3}) = -\frac{5}{7}$.
2. Show that $-\frac{1}{2} \times (-\frac{8}{9}) = \frac{4}{9}$.
3. Show that $-\frac{8}{10} \times \frac{15}{16} = -\frac{3}{4}$.
4. Show that $-\frac{10}{14} \times (-\frac{49}{18}) = \frac{35}{18}$.

Addition of Rational Numbers

On the number line below, what number is represented by the segment from $\frac{0}{16}$ to $\frac{3}{16}$? By the segment from $\frac{3}{16}$ to $\frac{12}{16}$? How is a single fraction for $\frac{3}{16} + \frac{9}{16}$ found on the number line?



What number is named by $\frac{3}{16} + \frac{9}{16}$? What do $\frac{3}{16}$ and $\frac{9}{16}$ have in common? The sum of $\frac{3}{16}$ and $\frac{9}{16}$ can be found as shown below.

$$\frac{3}{16} + \frac{9}{16} = \frac{3+9}{16} = \frac{12}{16} \text{ or } \frac{3}{4}$$

This method of finding the sum of rational numbers which have the same denominator can be stated as follows.

If $\frac{a}{c}$ and $\frac{b}{c}$ represent two rational numbers, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Oral Explain how the sum is found in each example below.

Written Copy. Find a single fraction in simplest form for each sum.

$$\begin{aligned} 1. \quad n &= \frac{7}{4} + \frac{11}{4} \\ &= \frac{7+11}{4} \\ &= \frac{18}{4} \text{ or } \frac{9}{2} \end{aligned}$$

$$\begin{aligned} n &= \frac{9}{14} + \frac{2}{14} \\ &= \frac{9+2}{14} \\ &= \frac{11}{14} \end{aligned}$$

$$1. \quad \frac{2}{5} + \frac{3}{5} \qquad \frac{2}{3} + \frac{3}{3} \qquad \frac{1}{6} + \frac{3}{6}$$

$$2. \quad \frac{3}{6} + \frac{1}{6} \qquad \frac{1}{4} + \frac{5}{4} \qquad \frac{3}{8} + \frac{5}{8}$$

$$3. \quad \frac{5}{9} + \frac{2}{9} \qquad \frac{4}{6} + \frac{3}{6} \qquad \frac{7}{12} + \frac{5}{12}$$

$$4. \quad \frac{8}{16} + \frac{7}{16} \qquad \frac{3}{10} + \frac{4}{10} \qquad \frac{1}{3} + \frac{2}{3}$$

$$5. \quad \frac{1}{10} + \frac{7}{10} \qquad \frac{7}{12} + \frac{5}{12} \qquad \frac{3}{12} + \frac{8}{12}$$

$$6. \quad \frac{1}{4} + \frac{3}{4} \qquad \frac{4}{6} + \frac{3}{6} \qquad \frac{7}{16} + \frac{5}{16}$$

Mentally solve each open sentence below.

$$2. \quad \frac{3}{8} + \frac{2}{8} = n \qquad \frac{2}{5} + \frac{1}{5} = n \qquad \frac{3}{11} + \frac{4}{11} = n$$

$$3. \quad \frac{7}{20} + \frac{10}{20} = n \qquad \frac{6}{2} + \frac{2}{2} = n \qquad \frac{5}{13} + \frac{6}{13} = n$$

Mixed Numerals

Is the numerator or the denominator of $\frac{17}{12}$ greater? You can express any positive rational number like $\frac{17}{12}$ as a sum of a positive whole number and a positive rational number less than 1.

$$\frac{17}{12} = \frac{12+5}{12} = \frac{12}{12} + \frac{5}{12} = 1 + \frac{5}{12}$$

Why is 17 renamed as 12+5 instead of 11+6 or 10+7 or as some other sum? How would you explain that $\frac{12+5}{12} = \frac{12}{12} + \frac{5}{12}$?

You can then express $1 + \frac{5}{12}$ as $1\frac{5}{12}$. Numerals like $1\frac{5}{12}$ are called **mixed numerals**. The numeral $1\frac{5}{12}$ is read *one and five twelfths*. How would you read $5\frac{7}{9}$?

In $3\frac{5}{4}$, does the fraction name a number less than 1? In $7\frac{6}{10}$, is the fraction in simplest form? In $13\frac{2}{5}$, does the fraction name a number less than 1? Is it in simplest form? We say $13\frac{2}{5}$ is in simplest form.

A mixed numeral is in simplest form if the fraction is in simplest form and names a positive rational number less than 1.

Oral Read each numeral below. Tell whether or not it is in simplest form. If it is not in simplest form, tell why not and how you could change it to simplest form.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$2\frac{3}{8}$	$4\frac{4}{8}$	$7\frac{3}{4}$	$17\frac{4}{6}$
2.	$6\frac{7}{5}$	$8\frac{9}{4}$	$3\frac{12}{8}$	$23\frac{15}{12}$
3.	$9\frac{1}{2}$	$5\frac{5}{10}$	$4\frac{4}{3}$	$11\frac{8}{2}$
4.	$1\frac{9}{12}$	$2\frac{7}{9}$	$6\frac{2}{6}$	$42\frac{10}{8}$

Written Copy. Change each numeral to a mixed numeral in simplest form.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$\frac{7}{5}$	$\frac{5}{4}$	$\frac{16}{6}$	$\frac{12}{7}$
2.	$\frac{11}{2}$	$\frac{32}{6}$	$\frac{15}{4}$	$\frac{18}{15}$
3.	$\frac{21}{14}$	$\frac{17}{8}$	$\frac{12}{9}$	$\frac{17}{10}$
4.	$3\frac{5}{10}$	$1\frac{9}{12}$	$2\frac{8}{7}$	$15\frac{8}{5}$
5.	$2\frac{9}{4}$	$4\frac{6}{8}$	$6\frac{15}{9}$	$29\frac{12}{18}$
6.	$7\frac{5}{2}$	$3\frac{8}{10}$	$5\frac{24}{18}$	$32\frac{10}{15}$

Multiplication Involving Mixed Numerals

A $3 \times (4+5) = (3 \times 4) + (3 \times 5)$

B $+2 \times (-3 + 4) = (+2 \times -3) + (+2 \times +4)$

What property of multiplication and addition of whole numbers is illustrated by sentence A? Is the same property illustrated with integers in sentence B?

The distributive property of multiplication over addition also holds for rational numbers. You can use the distributive property as shown below to find the product when one of two factors is named by a mixed numeral.

$n = 5 \times 3\frac{1}{4}$		$m = 3\frac{1}{3} \times 7$
$= 5 \times (3 + \frac{1}{4})$	What is renamed here?	$= (3 + \frac{1}{3}) \times 7$
$= (5 \times 3) + (5 \times \frac{1}{4})$	What property is used here?	$= (3 \times 7) + (\frac{1}{3} \times 7)$
$= 15 + \frac{5}{4}$	How is this obtained?	$= 21 + \frac{7}{3}$
$= 15\frac{5}{4}$	Are these in simplest form?	$= 21\frac{7}{3}$
$= 16\frac{1}{4}$	How is this obtained?	$= 23\frac{1}{3}$

You can also compute $5 \times 3\frac{1}{4}$ and $3\frac{1}{3} \times 7$ by naming both factors as fractions.

$$\begin{aligned}
 5 \times 3\frac{1}{4} &= \frac{5}{1} \times \frac{13}{4} \\
 &= \frac{5 \times 13}{4} \\
 &= \frac{65}{4} \text{ or } 16\frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 3\frac{1}{3} \times 7 &= \frac{10}{3} \times \frac{7}{1} \\
 &= \frac{10 \times 7}{3} \\
 &= \frac{70}{3} \text{ or } 23\frac{1}{3}
 \end{aligned}$$

To find the product when each factor is named as a mixed numeral it is more convenient if the factors are not named as mixed numerals but as fractions. Study how the product of $3\frac{1}{2}$ and $2\frac{1}{4}$ is found in simplest form below.

$3\frac{1}{2} \times 2\frac{1}{4} = \frac{7}{2} \times \frac{9}{4}$	
$= \frac{7 \times 9}{2 \times 4}$	How is $\frac{7 \times 9}{2 \times 4}$ obtained from $3\frac{1}{2} \times 2\frac{1}{4}$?
$= \frac{63}{8}$	How is this obtained?
$= \frac{56+7}{8}$	Why is 63 named as $56+7$?
$= \frac{56}{8} + \frac{7}{8}$	
$= 7 + \frac{7}{8} \text{ or } 7\frac{7}{8}$	Is $7\frac{7}{8}$ in simplest form?

Oral Explain how to express each product below in simplest form.

$$\begin{aligned} 1. \quad 4 \times 6\frac{1}{4} &= 4 \times \left(6 + \frac{1}{4}\right) \\ &= (4 \times 6) + \left(4 \times \frac{1}{4}\right) \\ &= 24 + 1 \text{ or } 25 \end{aligned}$$

$$\begin{aligned} 2. \quad 3\frac{5}{6} \times 4 &= \frac{23}{6} \times \frac{4}{1} \\ &= \frac{23}{\cancel{6}^3} \times \frac{\cancel{4}^2}{1} \\ &= \frac{46}{3} \text{ or } 15\frac{1}{3} \end{aligned}$$

$$\begin{aligned} 3. \quad 3\frac{2}{3} \times 2\frac{1}{4} &= \frac{11}{3} \times \frac{9}{4} \\ &= \frac{11}{\cancel{3}^1} \times \frac{\cancel{9}^3}{4} \\ &= \frac{33}{4} \text{ or } 8\frac{1}{4} \end{aligned}$$

State a fraction in simplest form for each mixed numeral below.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
4. $1\frac{1}{2}$	$2\frac{1}{4}$	$3\frac{2}{3}$	$4\frac{2}{3}$
5. $2\frac{3}{4}$	$1\frac{3}{5}$	$6\frac{3}{10}$	$3\frac{7}{8}$

State a mixed numeral in simplest form for each fraction below.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
6. $\frac{12}{8}$	$\frac{9}{6}$	$\frac{10}{6}$	$\frac{8}{6}$
7. $\frac{11}{5}$	$\frac{21}{4}$	$\frac{7}{2}$	$\frac{8}{3}$

Written Copy. Use the distributive property to express each product in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $6 \times 2\frac{3}{4}$	$7 \times 8\frac{1}{3}$	$12 \times 7\frac{1}{2}$
2. $8\frac{1}{2} \times 5$	$9\frac{1}{4} \times 6$	$5\frac{2}{3} \times 9$
3. $4 \times 2\frac{5}{6}$	$3\frac{1}{4} \times 7$	$6 \times 3\frac{7}{8}$
4. $8\frac{1}{2} \times 12$	$5 \times 6\frac{4}{5}$	$12\frac{1}{3} \times 8$
5. $8 \times 11\frac{1}{3}$	$15\frac{1}{3} \times 12$	$5 \times 3\frac{7}{10}$

Copy. Express each product in simplest form by the method shown in *Oral 3*.

<i>a</i>	<i>b</i>	<i>c</i>
6. $4\frac{1}{2} \times 1\frac{3}{4}$	$1\frac{2}{3} \times 2\frac{1}{4}$	$2\frac{1}{2} \times 2\frac{5}{6}$
7. $2\frac{1}{3} \times 6\frac{1}{2}$	$3\frac{1}{5} \times 4\frac{2}{3}$	$2\frac{1}{6} \times 2\frac{1}{3}$
8. $4\frac{1}{2} \times 3\frac{1}{4}$	$2\frac{1}{2} \times 8\frac{1}{3}$	$2\frac{1}{3} \times 8\frac{1}{2}$
9. $2\frac{1}{3} \times 3\frac{5}{8}$	$2\frac{1}{3} \times 4\frac{1}{8}$	$5\frac{2}{3} \times 3\frac{1}{4}$
10. $2\frac{1}{3} \times 2\frac{1}{3}$	$2\frac{1}{8} \times 1\frac{3}{4}$	$4\frac{1}{6} \times 2\frac{1}{4}$

Another way You can find the product of $3\frac{1}{2}$ and $2\frac{1}{4}$ by using the distributive property of multiplication over addition.

$$\begin{aligned} 3\frac{1}{2} \times 2\frac{1}{4} &= \left(3 + \frac{1}{2}\right) \times \left(2 + \frac{1}{4}\right) \\ &= 3\left(2 + \frac{1}{4}\right) + \frac{1}{2}\left(2 + \frac{1}{4}\right) \\ &= (3 \times 2) + \left(3 \times \frac{1}{4}\right) + \left(\frac{1}{2} \times 2\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) \\ &= 6 + \frac{3}{4} + 1 + \frac{1}{8} \text{ or } 7\frac{7}{8} \end{aligned}$$

Use this method to compute in *Written 9–10*.

Division of Rational Numbers

A $7 \div 1$ or $\frac{7}{1}$

B $16 \div 1$ or $\frac{16}{1}$

C $\frac{4}{7} \div 1$ or $\frac{\frac{4}{7}}{1}$

What number is named by the expressions in A? In B? What can you say about the quotient when the divisor is 1 and the dividend is any number? Since this pattern also holds for rational numbers, what number is named by the expressions in C?

Think of changing the quotient $\frac{3}{5} \div \frac{4}{7}$ to simplest form. You can express the division as a fraction.

$$\frac{\frac{3}{5} \div \frac{4}{7}}{1} = \frac{\frac{3}{5}}{\frac{4}{7}}$$

In order to obtain a divisor of 1, you can multiply both numerator and denominator by the same number. What number can you multiply $\frac{4}{7}$ by to obtain a product of 1?

Since $\frac{4}{7} \times \frac{7}{4} = 1$, we call $\frac{4}{7}$ and $\frac{7}{4}$ **reciprocals** of each other. Any two numbers whose product is 1 are called *reciprocals* of each other.

Then you can proceed by multiplying both numerator and denominator by $\frac{7}{4}$.

$$\frac{\frac{3}{5} \div \frac{4}{7}}{1} = \frac{\frac{3}{5}}{\frac{4}{7}} = \frac{\frac{3}{5} \times \frac{7}{4}}{\frac{4}{7} \times \frac{7}{4}} = \frac{\frac{3}{5} \times \frac{7}{4}}{1} = \frac{3}{5} \times \frac{7}{4} \text{ or } \frac{21}{20}$$

Notice that $\frac{3}{5} \div \frac{4}{7}$ is equal to $\frac{3}{5} \times \frac{7}{4}$. How are these expressions alike? How are they different?

Dividing by a rational number is equivalent to multiplying by its reciprocal.

If $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers and $c \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

What is the reciprocal of 5 or $\frac{5}{1}$? How can you change $\frac{2}{3} \div 5$ to simplest form?

Oral Explain how each quotient below is changed to simplest form.

$$\begin{aligned}
 a \\
 1. \quad \frac{3}{4} \div \frac{5}{7} &= \frac{\frac{3}{4} \times \frac{7}{5}}{\frac{5}{7} \times \frac{7}{5}} \\
 &= \frac{\frac{3}{4} \times \frac{7}{5}}{1} \\
 &= \frac{3}{4} \times \frac{7}{5} \\
 &= \frac{21}{20} \text{ or } 1\frac{1}{20}
 \end{aligned}$$

$$\begin{aligned}
 b \\
 \frac{7}{12} \div \frac{5}{6} &= \frac{\frac{7}{12} \times \frac{6}{5}}{\frac{5}{6} \times \frac{6}{5}} \\
 &= \frac{\frac{7}{12} \times \frac{6}{5}}{1} \\
 &= \frac{7}{12} \times \frac{1}{5} \\
 &= \frac{7}{60}
 \end{aligned}$$

Written Copy. Change each quotient to simplest form.

a	b	c
1. $\frac{3}{4} \div \frac{3}{7}$	$\frac{5}{12} \div \frac{5}{18}$	$\frac{4}{7} \div \frac{1}{8}$
2. $\frac{9}{16} \div \frac{1}{4}$	$\frac{2}{9} \div \frac{2}{3}$	$\frac{4}{15} \div \frac{5}{8}$
3. $\frac{7}{12} \div \frac{3}{7}$	$\frac{15}{16} \div \frac{1}{2}$	$\frac{12}{13} \div \frac{1}{3}$
4. $\frac{9}{10} \div \frac{6}{7}$	$\frac{1}{5} \div \frac{2}{3}$	$\frac{7}{16} \div \frac{9}{10}$
5. $\frac{3}{16} \div \frac{4}{5}$	$\frac{7}{12} \div \frac{5}{6}$	$\frac{5}{9} \div \frac{3}{4}$
6. $\frac{5}{7} \div \frac{3}{5}$	$\frac{7}{11} \div \frac{1}{3}$	$\frac{9}{10} \div \frac{4}{10}$

Tell the reciprocal of each number named below.

a	b	c	d
2. $\frac{2}{3}$	$\frac{5}{6}$	$\frac{9}{7}$	6
3. $\frac{5}{4}$	$\frac{3}{10}$	1	$\frac{7}{12}$

For each division numeral below, tell the multiplication numeral you could use in changing to simplest form.

a	b	c
4. $\frac{1}{2} \div \frac{2}{7}$	$\frac{5}{16} \div \frac{2}{5}$	$2 \div \frac{3}{5}$
5. $\frac{10}{21} \div 8$	$\frac{3}{4} \div \frac{3}{4}$	$0 \div \frac{4}{7}$

Give a reason for the truth of each sentence below.

a	b
6. $\frac{3}{5} \div 1 = \frac{3}{5}$	$\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \times \frac{7}{3}$
7. $\frac{4}{9} \times \frac{9}{4} = 1$	$\frac{\frac{2}{3} \times \frac{3}{4}}{1} = \frac{2}{3} \times \frac{3}{4}$

Tell why To find the reciprocal of 7, express 7 as $\frac{7}{1}$ and then record the reciprocal as $\frac{1}{7}$. Why does 0 not have a reciprocal?

Can you do this? Solve each equation mentally.

a	b
1. $\frac{3}{4} \times n = 1$	$\frac{5}{8} \div a = \frac{5}{8} \times \frac{2}{3}$
2. $0 \div \frac{5}{7} = c$	$d \div \frac{7}{11} = 1$

Solving Problems

A contractor resurfaced $\frac{3}{4}$ mile of road in 5 days by resurfacing the same amount each day. What part of a mile of road was resurfaced each day?

Was the piece of road separated into parts of the same size? What operation on numbers does this suggest? What open division sentence can you write for the problem?

$$\frac{3}{4} \div 5 = \frac{3}{4} \div \frac{5}{1} = \frac{3}{4} \times \frac{1}{5} = \frac{3}{20}$$

What part of a mile of road was resurfaced each day?

A certain small appliance can be assembled by a worker in $\frac{5}{6}$ of an hour. How many of the appliances could be assembled by a worker in $2\frac{1}{2}$ hours?

What open sentence can you write for the problem? Why would you choose a division sentence? To solve $2\frac{1}{2} \div \frac{5}{6} = n$ you can name $2\frac{1}{2}$ as a fraction and perform the division as shown below.

$$n = 2\frac{1}{2} \div \frac{5}{6}$$

$$= \frac{5}{2} \div \frac{5}{6}$$

Rename $2\frac{1}{2}$ as a fraction.

$$= \frac{5}{2} \times \frac{6}{5}$$

Why?

$$= \frac{\cancel{5}^1}{2} \times \frac{\cancel{6}_3}{\cancel{5}_1}$$

Explain this step.

$$= \frac{1 \times 3}{1 \times 1} \text{ or } 3$$

What is the answer to the problem?

When mixed numerals or numerals for whole numbers occur in a division expression, change them to fractions and use the division pattern to find the quotient.

Oral Explain how each quotient below is changed to simplest form.

<i>a</i>	<i>b</i>
1. $n = \frac{7}{10} \div 4$ $= \frac{7}{10} \div \frac{4}{1}$ $= \frac{7}{10} \times \frac{1}{4}$ $= \frac{7 \times 1}{4 \times 10}$ $= \frac{7}{40}$	$n = 5 \div \frac{3}{8}$ $= \frac{5}{1} \div \frac{3}{8}$ $= \frac{5}{1} \times \frac{8}{3}$ $= \frac{5 \times 8}{1 \times 3}$ $= \frac{40}{3}$ or $13\frac{1}{3}$
2. $n = \frac{5}{8} \div 1\frac{2}{3}$ $= \frac{5}{8} \div \frac{5}{3}$ $= \frac{5}{8} \times \frac{3}{5}$ $= \frac{1}{8} \times \frac{3}{1}$ $= \frac{1 \times 3}{8 \times 1}$ or $\frac{3}{8}$	$n = 3\frac{1}{2} \div 2\frac{1}{4}$ $= \frac{7}{2} \div \frac{9}{4}$ $= \frac{7}{2} \times \frac{4}{9}$ $= \frac{7}{1} \times \frac{2}{9}$ $= \frac{7 \times 2}{1 \times 9}$ $= \frac{14}{9}$ or $1\frac{5}{9}$

Change each numeral below to a fraction in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
3. $2\frac{3}{7}$	$4\frac{3}{5}$	$8\frac{1}{5}$	$7\frac{2}{3}$
4. $2\frac{1}{3}$	$3\frac{1}{2}$	$4\frac{2}{3}$	$1\frac{5}{6}$
5. $5\frac{2}{3}$	$24\frac{1}{2}$	$32\frac{1}{3}$	$43\frac{1}{2}$
6. $1\frac{3}{8}$	$2\frac{3}{5}$	$7\frac{1}{2}$	$4\frac{2}{3}$
7. $12\frac{1}{3}$	$18\frac{1}{4}$	$16\frac{2}{5}$	$24\frac{5}{6}$

Written Copy. Change each quotient to simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $5 \div \frac{1}{3}$	$8 \div \frac{3}{4}$	$6 \div \frac{3}{4}$
2. $\frac{5}{6} \div 4$	$\frac{3}{8} \div 5$	$\frac{6}{7} \div 3$

3. $3\frac{1}{4} \div 5$	$8\frac{1}{3} \div 3$	$7\frac{1}{2} \div 4$
4. $8 \div 2\frac{1}{2}$	$9 \div 3\frac{1}{3}$	$10 \div 2\frac{2}{5}$
5. $1\frac{3}{4} \div \frac{7}{8}$	$2\frac{1}{2} \div \frac{2}{5}$	$2\frac{1}{4} \div \frac{2}{3}$
6. $2\frac{1}{5} \div \frac{3}{5}$	$2\frac{1}{4} \div \frac{3}{4}$	$3\frac{1}{3} \div \frac{7}{8}$
7. $3\frac{1}{2} \div 2\frac{1}{4}$	$6\frac{1}{2} \div 2\frac{2}{3}$	$3\frac{2}{3} \div 2\frac{1}{4}$
8. $6\frac{1}{8} \div 1\frac{3}{4}$	$3\frac{3}{4} \div 1\frac{1}{4}$	$7\frac{1}{3} \div 2\frac{3}{4}$
9. $2\frac{1}{2} \div 3\frac{3}{4}$	$3\frac{1}{8} \div 1\frac{2}{3}$	$8\frac{1}{4} \div 2\frac{3}{4}$
10. $1\frac{2}{3} \div 2\frac{1}{2}$	$1\frac{3}{4} \div 3\frac{1}{3}$	$5\frac{1}{4} \div 3\frac{1}{2}$

Write an open sentence for each problem below. Solve the open sentence. Then write an answer for each problem.

11. A merchant put $4\frac{1}{2}$ pounds of dried beans into 8 plastic bags. If each bag is to contain the same amount by weight, what is the weight of the beans in each bag?

12. Each candy bar weighs $2\frac{3}{4}$ ounces and the total weight of the candy bars in a box is 66 ounces. How many candy bars are there in the box?

13. A rope $13\frac{1}{2}$ feet long is to be cut in 9 parts of the same length. How long is each part?

14. A barrel contained $11\frac{1}{4}$ gallons of water. How many water jugs, holding $2\frac{1}{4}$ gallons each, can be filled with the water in the barrel?

Operations on Rational Numbers

Part 1 Copy. Change each numeral below to a fraction in simplest form.

	a	b	c	d	e
1.	$\frac{12}{16}$	$\frac{9}{12}$	$\frac{8}{12}$	$\frac{14}{24}$	$\frac{16}{24}$
2.	$\frac{8}{16}$	$\frac{5}{10}$	$\frac{3}{12}$	$\frac{24}{32}$	$\frac{6}{16}$
3.	$\frac{1}{8}$	$\frac{18}{24}$	$\frac{20}{24}$	$\frac{2}{12}$	$\frac{4}{14}$
4.	$\frac{9}{18}$	$\frac{8}{10}$	$\frac{5}{15}$	$\frac{4}{16}$	$\frac{6}{9}$

Part 2 Copy. Change each quotient to simplest form.

	a	b	c
1.	$\frac{3}{8} \div \frac{1}{2}$	$\frac{5}{6} \div \frac{1}{3}$	$\frac{9}{10} \div \frac{2}{3}$
2.	$\frac{9}{16} \div \frac{3}{4}$	$\frac{5}{9} \div \frac{5}{6}$	$\frac{4}{8} \div \frac{1}{3}$
3.	$4 \div \frac{3}{8}$	$5 \div \frac{3}{5}$	$7 \div \frac{2}{3}$
4.	$\frac{7}{8} \div 2$	$\frac{11}{16} \div 3$	$\frac{12}{13} \div 4$
5.	$4 \div 1\frac{1}{2}$	$6 \div 1\frac{3}{4}$	$7 \div 3\frac{1}{2}$

Part 3 Copy. Change each product to simplest form.

	a	b	c
1.	$\frac{1}{2} \times \frac{5}{8}$	$\frac{2}{3} \times \frac{1}{5}$	$\frac{5}{7} \times \frac{2}{3}$
2.	$\frac{5}{8} \times \frac{2}{3}$	$\frac{5}{6} \times \frac{3}{4}$	$\frac{7}{9} \times \frac{3}{4}$
3.	$4 \times \frac{5}{6}$	$5 \times \frac{7}{8}$	$6 \times \frac{2}{3}$
4.	$\frac{7}{8} \times 9$	$\frac{3}{4} \times 7$	$\frac{2}{5} \times 8$
5.	$4 \times 3\frac{1}{4}$	$5 \times 3\frac{1}{3}$	$7 \times 1\frac{3}{5}$

Part 4 Write an open sentence for each problem below. Solve the open sentence. Write an answer for the problem.

1. There is a pond halfway between Jack's house and town. If Jack lives $5\frac{3}{4}$ miles from town, how far is it from town to the pond?

2. Mr. Bond cut a rope which was $15\frac{3}{4}$ ft. long into 4 pieces of the same length. How long was each piece of rope?

3. A certain water pump can pump $12\frac{1}{2}$ gallons of water per minute. How many gallons of water can it pump in $4\frac{1}{2}$ minutes?

4. Mr. Sims owned $\frac{3}{4}$ interest in a grocery business. He sold $\frac{1}{2}$ of his share. What part of the business did he own after the sale?

5. Susan earned $2\frac{1}{10}$ dollars by baby sitting for $3\frac{1}{2}$ hours. What was the hourly rate?

6. On a test run, a racing car traveled $10\frac{5}{6}$ miles in 5 minutes. How many miles per minute was its average speed?

7. One gallon of water weighs about $8\frac{1}{3}$ pounds. What is the approximate weight of $17\frac{4}{5}$ gallons of water?

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Sentences like $2 \div 3 = r$ and $-5 \div 2 = s$ do not have solutions in the set of whole numbers nor the set of integers, but they do in the set of rational numbers. (107, 110)

2. The same properties of multiplication of integers exist for multiplication of rational numbers. (112, 114, 115)

3. You can use products of powers of prime factors to find the greatest common factor of two or more numbers. (116)

4. To express a fraction in simplest form, both the numerator and the denominator are divided by their greatest common factor. (118)

5. You can find a quotient of two rational numbers by finding the product of the dividend and the reciprocal of the divisor. (126)

Words to Know

1. Fraction (108)

2. Rational number (110)

3. Commutative and associative properties and the identity number of multiplication of rational numbers. (114, 115)

4. Greatest common factor, relatively prime (116)

5. Simplest form (118)

6. Mixed numerals (123)

Questions to Discuss

1. How can the product of $\frac{7}{8}$ and $\frac{3}{4}$ be named as a single fraction? (112)

2. How can you tell if a fraction is in simplest form? (118)

3. How would you change $\frac{3}{5} \times \frac{2}{3}$ to simplest form? (120)

4. How can you find the sum of rational numbers which have the same denominator? (122)

5. How can the division expression $\frac{2}{3} \div \frac{3}{4}$ be changed to a multiplication expression? (126)

Written Practice

Copy. Change each expression to simplest form.

	<i>a</i>	<i>b</i>	
1.	$\frac{5}{6} \times \frac{1}{2}$	$3 \times \frac{3}{4}$	(112)
2.	$5 \times 3\frac{1}{3}$	$3\frac{1}{4} \times 2\frac{1}{2}$	(124)
3.	$\frac{5}{9} \div \frac{1}{4}$	$\frac{5}{9} \div \frac{5}{18}$	(126)
4.	$2\frac{1}{2} \div 3$	$1\frac{3}{4} \div 3\frac{1}{3}$	(128)

Self-Evaluation

Part 1 Find the greatest common factor of each pair of numbers below.

<i>a</i>	<i>b</i>
1. 12 and 16	6 and 16
2. 8 and 18	16 and 14
3. 8 and 24	8 and 10
4. 16 and 20	24 and 12

Part 2 Copy. Express each number as a mixed numeral in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1. $\frac{11}{4}$	$\frac{15}{8}$	$\frac{7}{3}$	$\frac{9}{2}$
2. $\frac{10}{4}$	$\frac{12}{8}$	$\frac{14}{6}$	$\frac{9}{6}$
3. $7\frac{2}{4}$	$4\frac{9}{12}$	$5\frac{10}{12}$	$3\frac{8}{24}$
4. $2\frac{12}{8}$	$1\frac{15}{7}$	$3\frac{7}{2}$	$4\frac{6}{4}$

Part 3 Copy. Change each product to simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{3}{4} \times \frac{1}{2}$	$\frac{1}{3} \times \frac{5}{6}$	$\frac{7}{8} \times \frac{1}{2}$
2. $\frac{3}{4} \times \frac{4}{5}$	$\frac{6}{7} \times \frac{3}{8}$	$\frac{2}{3} \times \frac{3}{8}$
3. $\frac{5}{8} \times \frac{4}{5}$	$5 \times \frac{3}{8}$	$\frac{5}{6} \times 7$
4. $\frac{1}{7} \times \frac{7}{9}$	$\frac{2}{3} \times 9$	$8 \times \frac{3}{4}$
5. $3\frac{1}{4} \times 2\frac{1}{2}$	$1\frac{3}{4} \times 2\frac{1}{3}$	$1\frac{1}{2} \times 3\frac{1}{3}$
6. $4\frac{1}{6} \times 1\frac{3}{5}$	$2\frac{1}{8} \times 2\frac{2}{3}$	$4\frac{2}{3} \times 1\frac{1}{2}$

Part 4 Copy. Change each quotient to simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{3}{4} \div \frac{1}{2}$	$\frac{7}{16} \div \frac{2}{3}$	$\frac{9}{10} \div \frac{2}{3}$
2. $\frac{11}{24} \div \frac{3}{4}$	$\frac{7}{8} \div \frac{1}{3}$	$\frac{7}{9} \div \frac{5}{6}$
3. $\frac{13}{16} \div \frac{3}{4}$	$\frac{7}{8} \div 4$	$4 \div \frac{5}{6}$
4. $\frac{7}{8} \div \frac{2}{3}$	$5 \div \frac{2}{3}$	$6 \div \frac{7}{8}$
5. $3\frac{1}{4} \div 2\frac{1}{2}$	$1\frac{3}{4} \div 2\frac{1}{6}$	$1\frac{7}{8} \div 4\frac{1}{2}$
6. $9\frac{5}{6} \div 2\frac{1}{2}$	$4\frac{1}{8} \div 1\frac{3}{4}$	$5\frac{2}{3} \div 1\frac{1}{3}$

Part 5 Write an open sentence for each problem. Solve the open sentence. Answer the problem.

1. A checkerboard is made up of 8 rows of $1\frac{1}{2}$ -inch squares. How long is each side of the checkerboard?

2. How many boxes $3\frac{1}{2}$ feet wide can be placed along a wall of a warehouse, if the wall is 56 feet long?

3. A wagon is $1\frac{1}{3}$ yards long. If its width is half of its length, how wide is the wagon?

4. If a car averaged $57\frac{1}{2}$ miles per hour for $3\frac{1}{2}$ hours, how far did the car travel during that period?

5. How many pieces of rope $2\frac{2}{3}$ ft. long can be cut from a piece of rope $18\frac{2}{3}$ ft. long?

Chapter 6

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

Renaming Rational Numbers.

$$\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \dots$$

Each of the fractions above is a name for the identity number of multiplication of rational numbers. You can use these names for *one* to find other fractions for a rational number like $\frac{3}{4}$.

$$\begin{aligned} \frac{3}{4} \times 1 &= \frac{3}{4} \times \frac{2}{2} \\ &= \frac{3 \times 2}{4 \times 2} \\ &= \frac{6}{8} \end{aligned}$$

$$\begin{aligned} \frac{3}{4} \times 1 &= \frac{3}{4} \times \frac{3}{3} \\ &= \frac{3 \times 3}{4 \times 3} \\ &= \frac{9}{12} \end{aligned}$$

$$\begin{aligned} \frac{3}{4} \times 1 &= \frac{3}{4} \times \frac{4}{4} \\ &= \frac{3 \times 4}{4 \times 4} \\ &= \frac{12}{16} \end{aligned}$$

$$\begin{aligned} \frac{3}{4} \times 1 &= \frac{3}{4} \times \frac{5}{5} \\ &= \frac{3 \times 5}{4 \times 5} \\ &= \frac{15}{20} \end{aligned}$$

Is the effect of multiplying $\frac{3}{4}$ by 1 the same as multiplying both the numerator and the denominator by the same number? What is that number when $\frac{3}{4}$ is changed to $\frac{6}{8}$? To $\frac{9}{12}$? To $\frac{12}{16}$? To $\frac{15}{20}$? This method of renaming can be stated as follows.

If $\frac{a}{b}$ represents a rational number and k represents a natural number, then

$$\frac{a}{b} = \frac{a \times k}{b \times k} \quad \text{or} \quad \frac{a}{b} = \frac{ak}{bk}.$$

Oral Tell the next 3 fractions in each row. Then tell how you obtained each of those fractions.

1. $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \text{---}, \text{---}, \text{---}$

2. $\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \text{---}, \text{---}, \text{---}$

3. $\frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \text{---}, \text{---}, \text{---}$

Written Do the following.

1-3. Copy *Oral* 1-3. Write the next three fractions in each list.

4. Use $\frac{7}{7}, \frac{11}{11}$, and $\frac{21}{21}$ as names for *one* to write three more names for each of the following.

$$\frac{3}{7}, \frac{5}{8}, \frac{9}{12}, \frac{12}{17}$$

Equality of Rational Numbers

You have already discovered that every rational number can be named by many fractions.

$$\frac{3}{4} = \frac{6}{8}, \quad \frac{3}{4} = \frac{9}{12}, \quad \frac{3}{4} = \frac{12}{16}, \quad \frac{3}{4} = \frac{15}{20}, \dots$$

Since $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$, \dots all name the same number, they are called **equivalent fractions**.

Fractions that name the same number are called *equivalent fractions*.

By studying the following examples you can discover a method for telling if two fractions are equivalent.

$$\frac{3}{4} = \frac{6}{8} \xrightarrow{4 \times 6} 3 \times 8$$

$$\frac{3}{4} = \frac{9}{12} \xrightarrow{4 \times 9} 3 \times 12$$

$$\frac{3}{4} = \frac{12}{16} \xrightarrow{4 \times 12} 3 \times 16$$

How are 4×6 and 3×8 obtained from $\frac{3}{4}$ and $\frac{6}{8}$? Do 4×6 and 3×8 name the same number? Does the pattern hold for $\frac{3}{4} = \frac{9}{12}$ and $\frac{3}{4} = \frac{12}{16}$? This pattern can also be shown as follows.

$$\text{If } \frac{3}{4} = \frac{6}{8}, \text{ then } 3 \times 8 = 4 \times 6.$$

Suppose you start with two multiplication numerals for the same number, like 3×6 and 2×9 . Could you reverse the procedure above to write two equivalent fractions?

$$3 \times 6 = 2 \times 9 \xrightarrow{3 \times 9} \frac{3}{2} = \frac{9}{6}$$

or

$$\text{If } 3 \times 6 = 2 \times 9, \text{ then } \frac{3}{2} = \frac{9}{6}.$$

The relationship between equivalent fractions can be stated as follows.

Let $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

$$\text{If } ad = bc, \text{ then } \frac{a}{b} = \frac{c}{d}.$$

The relationship between equivalent fractions can be used to solve any of the open sentences below.

$\frac{n}{5} = \frac{14}{35}$	$\frac{3}{n} = \frac{21}{56}$	$\frac{9}{4} = \frac{n}{12}$	$\frac{7}{11} = \frac{49}{n}$
$n \times 35 = 5 \times 14$	$3 \times 56 = n \times 21$	$9 \times 12 = 4n$	$7n = 11 \times 49$
$n \times 35 = 70$	$168 = n \times 21$	$108 = 4n$	$7n = 539$
$n = 2$	$8 = n$	$27 = n$	$n = 77$

Since multiplication is commutative, you can derive four different pairs of equivalent fractions from any expression like $3 \times 6 = 2 \times 9$.

$$3 \times 6 = 2 \times 9$$

$$\frac{3}{2} = \frac{9}{6}$$

$$6 \times 3 = 2 \times 9$$

$$\frac{6}{2} = \frac{9}{3}$$

$$3 \times 6 = 9 \times 2$$

$$\frac{3}{9} = \frac{2}{6}$$

$$6 \times 3 = 9 \times 2$$

$$\frac{6}{9} = \frac{2}{3}$$

Oral Tell whether $=$ or \neq should replace each \bullet to make each sentence below become true.

	a	b	c
1.	$\frac{2}{3} \bullet \frac{6}{9}$	$\frac{3}{4} \bullet \frac{8}{12}$	$\frac{5}{6} \bullet \frac{20}{24}$
2.	$\frac{7}{8} \bullet \frac{13}{16}$	$\frac{4}{7} \bullet \frac{8}{14}$	$\frac{1}{4} \bullet \frac{2}{6}$
3.	$\frac{1}{3} \bullet \frac{3}{10}$	$\frac{1}{5} \bullet \frac{4}{20}$	$\frac{5}{7} \bullet \frac{8}{11}$
4.	$\frac{6}{10} \bullet \frac{9}{15}$	$\frac{16}{9} \bullet \frac{9}{5}$	$\frac{7}{12} \bullet \frac{9}{15}$

Answer the questions below.

5. By commuting factors, how could you derive four pairs of equivalent fractions from $5 \times 9 = 3 \times 15$?

6. How would you solve the equation $\frac{9}{24} = \frac{21}{n}$?

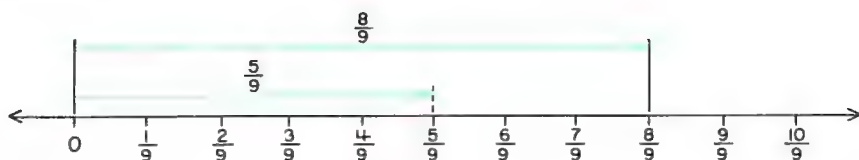
Written Copy. Use the pattern for equivalent fractions to determine which sentences are true.

	a	b	c
1.	$\frac{7}{8} = \frac{14}{16}$	$\frac{5}{6} = \frac{10}{12}$	$\frac{1}{6} = \frac{2}{12}$
2.	$\frac{4}{5} = \frac{8}{10}$	$\frac{5}{8} = \frac{10}{16}$	$\frac{3}{6} = \frac{2}{4}$
3.	$\frac{6}{21} = \frac{2}{7}$	$\frac{9}{12} = \frac{6}{8}$	$\frac{5}{10} = \frac{7}{14}$
4.	$\frac{15}{9} = \frac{10}{6}$	$\frac{5}{4} = \frac{20}{16}$	$\frac{5}{4} = \frac{30}{24}$

Copy. Solve each equation.

	a	b	c
5.	$\frac{2}{3} = \frac{n}{15}$	$\frac{5}{6} = \frac{n}{24}$	$\frac{19}{38} = \frac{n}{4}$
6.	$\frac{n}{3} = \frac{7}{21}$	$\frac{n}{12} = \frac{15}{9}$	$\frac{n}{6} = \frac{20}{8}$
7.	$\frac{6}{n} = \frac{9}{15}$	$\frac{8}{n} = \frac{4}{5}$	$\frac{3}{n} = \frac{6}{16}$
8.	$\frac{10}{15} = \frac{2}{n}$	$\frac{2}{16} = \frac{5}{n}$	$\frac{9}{24} = \frac{15}{n}$

Inequality of Rational Numbers



Which is greater: $\frac{5}{9}$ or $\frac{8}{9}$? How can you tell? Which has the greater numerator: $\frac{5}{9}$ or $\frac{8}{9}$?

If two rational numbers have the same denominator, the one with the greater numerator is the greater number.

Since $8 > 5$, you know that $\frac{8}{9} > \frac{5}{9}$.

Since $5 < 8$, you know that $\frac{5}{9} < \frac{8}{9}$.

Notice the results when the pattern for equivalent fractions is applied to non-equivalent fractions.

$$\frac{8}{9} > \frac{5}{9} \quad \begin{array}{c} \swarrow \quad \searrow \\ 8 \times 9 > 9 \times 5 \\ \nwarrow \quad \nearrow \end{array}$$

If $\frac{8}{9} > \frac{5}{9}$, then $8 \times 9 > 9 \times 5$.

$$8 \times 9 > 9 \times 5 \quad \begin{array}{c} \swarrow \quad \searrow \\ \frac{8}{9} > \frac{5}{9} \\ \nwarrow \quad \nearrow \end{array}$$

If $8 \times 9 > 9 \times 5$, then $\frac{8}{9} > \frac{5}{9}$.

Now let us try this idea on two rational numbers that are obviously not equal and have different denominators. For example, you know that $\frac{2}{5} < \frac{4}{3}$ since $\frac{2}{5} < 1$ and $\frac{4}{3} > 1$.

$$\frac{2}{5} < \frac{4}{3} \quad \begin{array}{c} \swarrow \quad \searrow \\ 2 \times 3 < 5 \times 4 \\ \nwarrow \quad \nearrow \end{array}$$

If $\frac{2}{5} < \frac{4}{3}$, then $2 \times 3 < 5 \times 4$.

$$2 \times 3 < 5 \times 4 \quad \begin{array}{c} \swarrow \quad \searrow \\ \frac{2}{5} < \frac{4}{3} \\ \nwarrow \quad \nearrow \end{array}$$

If $2 \times 3 < 5 \times 4$, then $\frac{2}{5} < \frac{4}{3}$.

This inequality relationship can be stated as follows.

Let $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers and let b and d represent positive integers.

If $\frac{a}{b} > \frac{c}{d}$, then $ad > bc$.

If $ad > bc$, then $\frac{a}{b} > \frac{c}{d}$.

If $\frac{a}{b} < \frac{c}{d}$, then $ad < bc$.

If $ad < bc$, then $\frac{a}{b} < \frac{c}{d}$.

Finally, let us determine which is greater: $\frac{7}{12}$ or $\frac{7}{10}$. What is alike about these two numbers? Which has the greater denominator?



Since $7 \times 10 < 12 \times 7$,
you know that $\frac{7}{12} < \frac{7}{10}$.

If two rational numbers have the same numerator, the one with the lesser denominator is the greater number.

Oral Tell which fraction in each pair below names the greater number. Tell how you decided.

- | a | b | c |
|--------------------------------|------------------------------|--------------------------------|
| 1. $\frac{7}{8}, \frac{5}{8}$ | $\frac{9}{16}, \frac{7}{16}$ | $\frac{7}{24}, \frac{11}{24}$ |
| 2. $\frac{3}{7}, \frac{3}{11}$ | $\frac{1}{5}, \frac{1}{6}$ | $\frac{17}{31}, \frac{17}{29}$ |
| 3. $\frac{7}{5}, \frac{3}{2}$ | $\frac{5}{8}, \frac{3}{5}$ | $\frac{7}{9}, \frac{10}{13}$ |

Tell how you could show that each of the following sentences is a true sentence.

- | a | b | c |
|---------------------------------|--------------------------------|--------------------------------|
| 4. $\frac{3}{8} < \frac{7}{16}$ | $\frac{7}{8} > \frac{4}{5}$ | $\frac{2}{3} < \frac{3}{4}$ |
| 5. $\frac{6}{7} > \frac{4}{5}$ | $\frac{5}{11} < \frac{8}{16}$ | $\frac{9}{16} > \frac{11}{24}$ |
| 6. $\frac{3}{8} < \frac{4}{9}$ | $\frac{7}{12} > \frac{13}{23}$ | $\frac{12}{5} < \frac{5}{2}$ |

State an inequality sentence like $\frac{2}{3} < \frac{7}{8}$ for each sentence below.

- | a | b |
|------------------------------|---------------------------|
| 7. $3 \times 2 < 4 \times 5$ | $7 \times 5 > 3 \times 4$ |
| 8. $5 \times 8 < 6 \times 7$ | $9 \times 6 > 7 \times 5$ |

Written Copy. Replace each \bullet by either $<$ or $>$ so that the resulting sentence is true.

- | a | b | c |
|--|--------------------------------------|--------------------------------------|
| 1. $\frac{4}{5} \bullet \frac{5}{6}$ | $\frac{5}{12} \bullet \frac{1}{4}$ | $\frac{1}{3} \bullet \frac{3}{8}$ |
| 2. $\frac{2}{3} \bullet \frac{7}{12}$ | $\frac{5}{7} \bullet \frac{6}{9}$ | $\frac{2}{5} \bullet \frac{3}{8}$ |
| 3. $\frac{4}{7} \bullet \frac{5}{8}$ | $\frac{5}{8} \bullet \frac{2}{3}$ | $\frac{7}{24} \bullet \frac{10}{35}$ |
| 4. $\frac{7}{8} \bullet \frac{11}{16}$ | $\frac{5}{6} \bullet \frac{7}{8}$ | $\frac{9}{4} \bullet \frac{26}{12}$ |
| 5. $\frac{9}{17} \bullet \frac{5}{9}$ | $\frac{11}{16} \bullet \frac{2}{3}$ | $\frac{6}{7} \bullet \frac{2}{3}$ |
| 6. $\frac{1}{4} \bullet \frac{5}{12}$ | $\frac{9}{10} \bullet \frac{11}{12}$ | $\frac{5}{16} \bullet \frac{7}{30}$ |
| 7. $\frac{11}{15} \bullet \frac{2}{3}$ | $\frac{5}{8} \bullet \frac{3}{5}$ | $\frac{5}{11} \bullet \frac{6}{13}$ |

Can you do this? By using only the inequality relationships stated in this lesson, show that the following sentence is true.

$$\text{If } \frac{2}{3} > \frac{7}{12}, \text{ then } \frac{12}{7} > \frac{3}{2}.$$

(Hint: If $\frac{2}{3} > \frac{7}{12}$, then $2 \times 12 > 3 \times 7$. Since multiplication is commutative, $12 \times 2 > 7 \times 3$.)

Least Common Denominator

Multiples of 3 = { . . . , 3, 6, 9, 12, 15, 18, 21, 24, 27, . . . }

Multiples of 4 = { . . . , 4, 8, 12, 16, 20, 24, 28, 32, . . . }

Do some numbers occur in both sets of multiples above? Which numbers? Those numbers are called **common multiples** of 3 and 4. What does *common multiple* mean?

Common multiples of 3 and 4 = {12, 24, 36, . . . }

Since 12 is the least positive number in the set of common multiples, it is called the **least common multiple** of 3 and 4. Is 12 divisible by 3? By 4? Is the least common multiple of two numbers always divisible by both numbers?

Another way of finding the least common multiple of two or more numbers is given below.

Find the least common multiple of 8 and 18.		Find the least common multiple of 4, 18, and 30.
$8 = 2 \times 2 \times 2 = 2^3$ $18 = 2 \times 3 \times 3 = 2^1 \times 3^2$	Express each number as a product of primes.	$4 = 2 \times 2 = 2^2$ $18 = 2 \times 3 \times 3 = 2^1 \times 3^2$ $30 = 2 \times 3 \times 5 = 2^1 \times 3^1 \times 5^1$
$2^3 \times 3^2$	Select the greatest power of every prime factor.	$2^2 \times 3^2 \times 5^1$
$2^3 \times 3^2 = 8 \times 9$ $= 72$	The product of the greatest powers of the prime factors is the least common multiple.	$2^2 \times 3^2 \times 5^1 = 4 \times 9 \times 5$ $= 36 \times 5$ $= 180$

Think of finding the least common multiple of 5 and 15. Is 15 a multiple of 5? What is their least common multiple?

Think of finding the least common multiples of 5 and 7. Do 5 and 7 have any common factors? Are they relatively prime? How would you find their least common multiple?

Oral Explain how the least common multiple is found for each set of numbers below.

6 and 8	4, 6, and 8
$6 = 2 \times 3$	$4 = 2^2$
$8 = 2^3$	$6 = 2 \times 3$
$2^3 \times 3 = 8 \times 3$	$8 = 2^3$
$= 24$	$2^3 \times 3 = 8 \times 3$
	$= 24$

Answer the questions below.

1. Are the least common multiples the same in both examples above?

2. Is 4 a factor of any of the other numbers in $\{4, 6, 8\}$?

3. In finding the least common multiple of 4, 6, and 8, is it necessary to consider the 4? Why or why not?

4. To find the least common multiple of 6, 8, and 12, you need only find the least common multiple of which two numbers?

Tell the least common multiple of the numbers in each set below. Tell how you decided on your answer.

	<i>a</i>	<i>b</i>	<i>c</i>
5.	4, 12	7, 28	3, 6, 8
6.	7, 11	5, 13	5, 10, 12
7.	8, 72	3, 7	3, 4, 12
8.	9, 12	5, 14	8, 12, 15

Written Find the least common multiple of the numbers in each set.

	<i>a</i>	<i>b</i>
1.	$\{6, 10\}$	$\{5, 6, 7\}$
2.	$\{9, 15\}$	$\{9, 15, 21\}$
3.	$\{12, 15\}$	$\{6, 18, 21\}$
4.	$\{12, 18\}$	$\{10, 12, 18\}$
5.	$\{10, 14\}$	$\{20, 40, 50\}$

For each sentence below, write *T* if it is true and write *F* if it is false.

6. The least common multiple of two prime numbers is their product.

7. The least common multiple of two even numbers is their product.

8. The least common multiple of two numbers is always greater than either of the numbers.

9. The least common multiple of two numbers is never less than either of the two numbers.

Can you do this? To find the least common multiple of 6, 15, and 21, notice that the numbers have a common factor of 3.

$$6 = 3 \times 2 \quad 15 = 3 \times 5 \quad 21 = 3 \times 7$$

Show that the least common multiple of 6, 15, and 21 is 3 times the least common multiple of 2, 5, and 7.

Addition of Rational Numbers

You have learned how to add positive rational numbers with the same denominator. To add rational numbers with different denominators, you merely rename the rational numbers so they have the same denominator. It is convenient, but not necessary, to use the least denominator possible for this purpose.

Think of adding $\frac{3}{4}$ and $\frac{5}{6}$. Are their denominators the same? To find the least denominator for both numbers, you can find the least common multiple of both denominators.

The **least common denominator** of two or more rational numbers is the least common multiple of their denominators.

You can add $\frac{3}{4}$ and $\frac{5}{6}$ in either way shown below.

$$\begin{aligned} \frac{3}{4} + \frac{5}{6} &= \frac{9}{12} + \frac{10}{12} \\ &= \frac{9+10}{12} \\ &= \frac{19}{12} \text{ or } 1\frac{7}{12} \end{aligned}$$

Since the least common denominator of 5 and 8 is 40, you can add $\frac{3}{5} + \frac{7}{8}$ in either way shown below.

$$\begin{aligned}\frac{3}{5} + \frac{7}{8} &= \frac{24}{40} + \frac{35}{40} \\ &= \frac{24+35}{40} \\ &= \frac{59}{40} \text{ or } 1\frac{19}{40}\end{aligned}$$

When adding numbers like $4\frac{1}{2}$ and $3\frac{5}{6}$, recall that $4\frac{1}{2}$ means $4 + \frac{1}{2}$ and that $3\frac{5}{6}$ means $3 + \frac{5}{6}$.

$$\begin{array}{r} 4\frac{1}{2} \\ + 3\frac{5}{6} \\ \hline \end{array} \longrightarrow \begin{array}{r} 4 + \frac{1}{2} \\ + 3 + \frac{5}{6} \\ \hline \end{array} \longrightarrow \begin{array}{r} 4 + \frac{3}{6} \\ + 3 + \frac{5}{6} \\ \hline 7 + \frac{8}{6} \end{array} \longrightarrow \begin{array}{r} 4\frac{3}{6} \\ + 3\frac{5}{6} \\ \hline 7\frac{8}{6} \text{ or } 8\frac{1}{3} \end{array}$$

Oral Study the following addition of rational numbers.

$$\begin{array}{r} \frac{7}{9} \\ + \frac{5}{6} \\ \hline \end{array} \longrightarrow \begin{array}{r} \frac{14}{18} \\ + \frac{15}{18} \\ \hline \frac{29}{18} \text{ or } 1\frac{11}{18} \end{array}$$

Answer questions 1–4 about the example above.

1. Why were both addends renamed?

2. How can you decide that 18 is the least common denominator?

3. How is $\frac{29}{18}$ obtained from $\frac{14}{18}$ and $\frac{15}{18}$?

4. How can you change $\frac{29}{18}$ to $1\frac{11}{18}$?

Think about adding $\frac{5}{7}$ and $\frac{9}{14}$. Answer the following questions.

5. Is it necessary to rename both addends? Why or why not?

6. What number would you use as the least common denominator?

7. Is $\frac{5}{7}$ greater than or less than $\frac{1}{2}$? Is $\frac{9}{14}$ greater than or less than $\frac{1}{2}$? Will the sum of $\frac{5}{7}$ and $\frac{9}{14}$ be greater than or less than 1? Why?

Explain how you would find the simplest numeral for each sum below.

a	b	c
8. $\frac{1}{3} + \frac{5}{6}$	$\frac{1}{3} + \frac{4}{5}$	$\frac{3}{4} + \frac{1}{6}$
9. $\frac{3}{5} + \frac{1}{4}$	$\frac{2}{5} + \frac{4}{15}$	$\frac{7}{10} + \frac{5}{6}$

Written Copy. Find the simplest numeral for each sum below. If a sum is greater than 1, name it by a mixed numeral.

a	b	c
1. $\frac{1}{3} + \frac{1}{6}$	$\frac{3}{5} + \frac{1}{2}$	$\frac{1}{2} + \frac{2}{7}$
2. $\frac{5}{7} + \frac{9}{14}$	$\frac{3}{8} + \frac{5}{12}$	$\frac{5}{7} + \frac{2}{3}$
3. $\frac{4}{11} + \frac{2}{3}$	$\frac{7}{8} + \frac{1}{6}$	$\frac{5}{12} + \frac{7}{18}$
4. $\frac{2}{5} + 3\frac{1}{4}$	$4\frac{1}{2} + \frac{5}{8}$	$7\frac{3}{8} + \frac{3}{4}$
5. $9\frac{5}{6} + 4\frac{1}{2}$	$7\frac{1}{3} + 3\frac{1}{4}$	$8\frac{5}{16} + 2\frac{5}{8}$
6. $\frac{7}{8} + \frac{2}{3}$	$1\frac{1}{4} + \frac{7}{12}$	$3\frac{2}{5} + 2\frac{1}{4}$
7. $\frac{3}{5} + \frac{7}{15}$	$2\frac{1}{2} + 5\frac{3}{4}$	$8\frac{7}{16} + 3\frac{5}{12}$
8. $\frac{9}{21} + \frac{8}{35}$	$15\frac{2}{3} + 9\frac{7}{10}$	$21\frac{7}{24} + 6\frac{3}{4}$

Can you do this? Study the following addition of rational numbers.

$$\frac{-3}{5} + \frac{-1}{3} = \frac{-9}{15} + \frac{-5}{15} = \frac{-9 + -5}{15} = \frac{-14}{15}$$

Find the simplest numeral for each sum below.

a	b	c
1. $\frac{-1}{5} + \frac{-1}{2}$	$\frac{-2}{3} + \frac{-1}{7}$	$\frac{-2}{9} + \frac{-1}{6}$
2. $\frac{-3}{8} + \frac{3}{4}$	$\frac{7}{12} + \frac{-9}{15}$	$\frac{-5}{4} + \frac{7}{18}$

Addition of Rational Numbers

$$\frac{1}{4} + \frac{5}{6} = \frac{3}{12} + \frac{10}{12} = \frac{3+10}{12} = \frac{13}{12} \text{ or } 1\frac{1}{12}$$

How is $\frac{1}{4}$ changed to $\frac{3}{12}$? How is $\frac{5}{6}$ changed to $\frac{10}{12}$? Let us show these changes in the computation below.

$$\frac{1}{4} + \frac{5}{6} = \frac{1 \times 3}{4 \times 3} + \frac{5 \times 2}{6 \times 2} = \frac{1 \times 3}{12} + \frac{5 \times 2}{12} = \frac{(1 \times 3) + (5 \times 2)}{12}$$

How would you express the denominator 12 as a product of prime factors?

$$\frac{(1 \times 3) + (5 \times 2)}{12} = \frac{(1 \times 3) + (5 \times 2)}{2 \times 2 \times 3}$$

By using the transitive property of equality, you can conclude the following equation. Notice how this equation can be obtained directly from $\frac{1}{4} + \frac{5}{6}$.

$$\frac{1}{4} + \frac{5}{6} = \frac{(1 \times 3) + (5 \times 2)}{2 \times 2 \times 3}$$

2 × 2 2 × 3

Prime factors of the denominators
Least common denominator

The 1 and the 5 in $(1 \times 3) + (5 \times 2)$ name the numerators of the original addends. Compare the prime factors of each original denominator with $2 \times 2 \times 3$. How can you determine the 3 and the 2 in $(1 \times 3) + (5 \times 2)$ from the original denominators and the least common denominator?

Then the sum of $\frac{1}{4}$ and $\frac{5}{6}$ can be shown as follows.

$$\frac{1}{4} + \frac{5}{6} = \frac{(1 \times 3) + (5 \times 2)}{2 \times 2 \times 3} = \frac{3+10}{12} = \frac{13}{12} \text{ or } 1\frac{1}{12}$$

Explain the following addition of rational numbers.

$$\frac{7}{12} + \frac{5}{18} = \frac{(7 \times 3) + (5 \times 2)}{2 \times 2 \times 3 \times 3} = \frac{21+10}{36} = \frac{31}{36}$$

This method of adding rational numbers can also be used to find the sum of three or more rational numbers.

$$\begin{aligned}
 \frac{2}{5} + \frac{5}{6} + \frac{1}{4} &= \frac{(2 \times 2 \times 2 \times 3) + (5 \times 2 \times 5) + (1 \times 3 \times 5)}{2 \times 2 \times 3 \times 5} \\
 &= \frac{(2 \times 12) + (5 \times 10) + (1 \times 15)}{60} \\
 &= \frac{24 + 50 + 15}{60} \\
 &= \frac{89}{60} \text{ or } 1\frac{29}{60}
 \end{aligned}$$

How is the denominator $2 \times 2 \times 3 \times 5$ determined? How is each addend of the numerator determined?

Oral Use the following addition of rational numbers to answer questions 1–5.

$$\begin{aligned}
 \frac{3}{8} + \frac{1}{6} + \frac{13}{25} &= \frac{(3 \times 75) + (1 \times 100) + (13 \times 24)}{2 \times 2 \times 2 \times 3 \times 5 \times 5} \\
 &= \frac{225 + 100 + 312}{600} \\
 &= \frac{637}{600} \text{ or } 1\frac{37}{600}
 \end{aligned}$$

1. How can you determine the denominator $2 \times 2 \times 2 \times 3 \times 5 \times 5$?

2. How can you determine the 3 and the 75 in (3×75) ?

3. How can you determine the 1 and the 100 in (1×100) ?

4. How can you determine the 13 and the 24 in (13×24) ?

5. How is $1\frac{37}{600}$ obtained from $\frac{637}{600}$? Is $1\frac{37}{600}$ in simplest form? How can you tell?

Written Copy. Find the simplest fraction or mixed numeral for each sum below.

- | | a | b |
|----|--|---|
| 1. | $\frac{7}{12} + \frac{1}{9} + \frac{3}{4}$ | $\frac{2}{3} + \frac{4}{5} + \frac{1}{6}$ |
| 2. | $\frac{7}{10} + \frac{2}{15} + \frac{1}{6}$ | $\frac{1}{4} + \frac{4}{25} + \frac{3}{10}$ |
| 3. | $\frac{3}{7} + \frac{1}{3} + \frac{2}{5}$ | $\frac{7}{2} + \frac{5}{4} + \frac{7}{6}$ |
| 4. | $3\frac{1}{2} + 1\frac{3}{5} + \frac{2}{3}$ | $\frac{5}{9} + 1\frac{4}{5} + 7\frac{1}{3}$ |
| 5. | $5\frac{1}{4} + 3\frac{3}{8} + 1\frac{5}{6}$ | $2\frac{4}{9} + \frac{1}{6} + \frac{3}{4}$ |

Can you do this? Find the sum for each row, column, and diagonal.

$5\frac{1}{3}$	$\frac{1}{12}$	$3\frac{5}{6}$
$1\frac{7}{12}$	$3\frac{1}{12}$	$4\frac{7}{12}$
$2\frac{1}{3}$	$6\frac{1}{12}$	$\frac{5}{6}$

Properties of Addition of Rational Numbers

$$\begin{aligned}\frac{3}{4} + \frac{1}{6} &= \frac{(3 \times 3) + (1 \times 2)}{2 \times 2 \times 3} \\ &= \frac{9+2}{12} \\ &= \frac{11}{12}\end{aligned}$$

$$\begin{aligned}\frac{1}{6} + \frac{3}{4} &= \frac{(1 \times 2) + (3 \times 3)}{2 \times 2 \times 3} \\ &= \frac{2+9}{12} \\ &= \frac{11}{12}\end{aligned}$$

Since $\frac{3}{4} + \frac{1}{6} = \frac{11}{12}$ and $\frac{1}{6} + \frac{3}{4} = \frac{11}{12}$, you use the transitive property of equality to conclude that $\frac{3}{4} + \frac{1}{6} = \frac{1}{6} + \frac{3}{4}$.

In finding the sum of two rational numbers, does changing the order of the addends change the sum? What property of addition of rational numbers does this illustrate?

If $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers, then

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}.$$

$$\begin{aligned}\left(\frac{2}{9} + \frac{1}{6}\right) + \frac{3}{4} &= \frac{4+3}{18} + \frac{3}{4} \\ &= \frac{7}{18} + \frac{3}{4} \\ &= \frac{14+27}{36} \\ &= \frac{41}{36}\end{aligned}$$

$$\begin{aligned}\frac{2}{9} + \left(\frac{1}{6} + \frac{3}{4}\right) &= \frac{2}{9} + \frac{2+9}{12} \\ &= \frac{2}{9} + \frac{11}{12} \\ &= \frac{8+33}{36} \\ &= \frac{41}{36}\end{aligned}$$

Since the result is $\frac{41}{36}$ in both examples above, why can you conclude that $\left(\frac{2}{9} + \frac{1}{6}\right) + \frac{3}{4} = \frac{2}{9} + \left(\frac{1}{6} + \frac{3}{4}\right)$? Does grouping the addends in different ways change the sum? What property of addition of rational numbers does this illustrate?

If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ represent rational numbers, then

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right).$$

$$\frac{3}{7} + \frac{0}{7} = \frac{3+0}{7} = \frac{3}{7}$$

$$\frac{0}{5} + \frac{4}{5} = \frac{0+4}{5} = \frac{4}{5}$$

$$\frac{3}{5} + \frac{0}{8} = \frac{(3 \times 8) + (0 \times 5)}{5 \times 8} = \frac{24+0}{40} = \frac{24}{40} = \frac{3}{5}$$

What number is named by $\frac{0}{7}$? By $\frac{0}{5}$? By $\frac{0}{8}$? If one of two rational numbers is zero, what is their sum?

Zero is called the identity number of addition of rational numbers.

Oral Tell which property of addition of rational numbers is illustrated by each sentence below.

- | a | b |
|--|---|
| 1. $\frac{3}{8} + 0 = \frac{3}{8}$ | $\frac{2}{3} + (\frac{1}{2} + \frac{5}{6}) = (\frac{2}{3} + \frac{1}{2}) + \frac{5}{6}$ |
| 2. $\frac{4}{7} + 1 = 1 + \frac{4}{7}$ | $\frac{2}{3} + (\frac{1}{2} + \frac{5}{6}) = (\frac{1}{2} + \frac{5}{6}) + \frac{2}{3}$ |
| 3. $0 + \frac{5}{9} = \frac{5}{9}$ | $(\frac{1}{8} + \frac{2}{3}) + \frac{7}{9} = \frac{1}{8} + (\frac{2}{3} + \frac{7}{9})$ |

Written Copy. Find the simplest numeral for each sum. Then compute the addends and add again.

- | a | b | c |
|----------------------------------|-------------------------------|--------------------------------|
| 1. $\frac{3}{8} + \frac{2}{3}$ | $\frac{2}{3} + \frac{1}{5}$ | $1\frac{2}{3} + \frac{1}{4}$ |
| 2. $\frac{11}{12} + \frac{1}{4}$ | $\frac{7}{16} + \frac{5}{6}$ | $\frac{3}{5} + 3\frac{3}{4}$ |
| 3. $\frac{4}{9} + \frac{5}{12}$ | $\frac{3}{10} + \frac{3}{4}$ | $7\frac{1}{2} + 5\frac{4}{5}$ |
| 4. $\frac{4}{7} + \frac{2}{5}$ | $\frac{7}{16} + \frac{5}{12}$ | $4\frac{7}{15} + 2\frac{2}{5}$ |

Copy each expression twice. Place () to associate the addends in two different ways. Find the simplest numeral for the sum in each case.

- | a | b |
|--|--|
| 5. $\frac{5}{6} + \frac{2}{3} + \frac{2}{9}$ | $\frac{1}{2} + \frac{3}{8} + \frac{5}{6}$ |
| 6. $\frac{5}{8} + \frac{1}{6} + \frac{2}{3}$ | $\frac{1}{8} + \frac{2}{3} + \frac{7}{9}$ |
| 7. $\frac{3}{4} + 2\frac{1}{5} + \frac{1}{3}$ | $3\frac{2}{5} + \frac{3}{9} + 1\frac{1}{6}$ |
| 8. $\frac{3}{16} + 1\frac{1}{10} + \frac{5}{12}$ | $4\frac{1}{8} + 5\frac{1}{2} + 2\frac{2}{3}$ |
| 9. $5\frac{2}{5} + 3\frac{2}{3} + \frac{5}{6}$ | $7\frac{3}{4} + 3\frac{5}{8} + 2\frac{1}{3}$ |

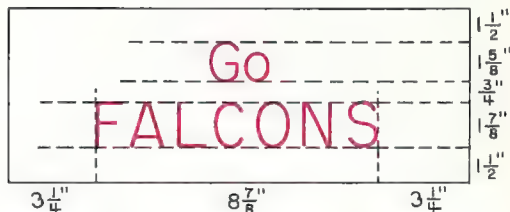
Write an open sentence for each problem. Solve the open sentence. Answer the problem.

10. For a science experiment, Jim mixed $2\frac{1}{4}$ ounces of oil, $1\frac{5}{8}$ ounces of glycerin, and $18\frac{1}{2}$ ounces of water. How many ounces were there in the final mixture?

11. One day Mr. Todd drove $16\frac{2}{3}$ miles from his home to his office. He then drove $6\frac{5}{8}$ miles to call on a customer and returned to the office. Finally, he drove home from the office. How many miles did he drive that day?

12. Three boys started mowing a lawn. Tom mowed $\frac{2}{5}$ of it; Ed mowed $\frac{1}{4}$ of it; and Dave mowed $\frac{1}{6}$ of it. How much of the lawn was mowed then?

13. Sally designed the sign below.



What will be the length and the width of the completed sign?

Can you do this? Solve this problem. When $\frac{3}{8}$ of the pupils in class are absent, 20 pupils are present. If $\frac{3}{4}$ of the class went on a field trip and the rest stayed at school, how many pupils stayed at school?

Subtraction of Rational Numbers

You have discovered the following about rational numbers.

$$\frac{-5}{+8} = -5 \div +8 = -(5 \div 8) = -\left(\frac{5}{8}\right) = -\frac{5}{8}$$

Therefore, $-\frac{5}{8} = \frac{-5}{+8}$ or $\frac{-5}{8}$.

Every negative rational number can be expressed so that the denominator is a positive integer.

When subtracting integers, you discovered the following relationship between addition and subtraction.

$$a - +b = a + -b$$

$$7 - 5 = 7 + -5 = 2$$

$$-a - -b = -a + +b$$

$$-8 - -3 = -8 + +3 = -5$$

This relationship also holds for subtraction of rational numbers.

$$\frac{8}{9} - \frac{5}{9} = \frac{8}{9} + \left(-\frac{5}{9}\right) = \frac{8}{9} + \frac{-5}{9} = \frac{8 + -5}{9} = \frac{3}{9} \text{ or } \frac{1}{3}$$

$$\frac{1}{4} - \frac{3}{5} = \frac{1}{4} + \left(-\frac{3}{5}\right) = \frac{1}{4} + \frac{-3}{5} = \frac{5}{20} + \frac{-12}{20} = \frac{5 + -12}{20} = \frac{-7}{20}$$

In a subtraction like $\frac{8}{9} - \frac{5}{9}$, where the number being subtracted is less than the number subtracted from, you can complete the computation without using negative numbers.

$$\begin{aligned} \frac{8}{9} - \frac{5}{9} &= \frac{8-5}{9} \\ &= \frac{3}{9} \text{ or } \frac{1}{3} \end{aligned}$$

$$\begin{array}{r} \frac{8}{9} \\ -\frac{5}{9} \\ \hline \frac{3}{9} \text{ or } \frac{1}{3} \end{array}$$

This method can be stated as follows.

If $\frac{a}{b}$ and $\frac{c}{b}$ represent rational numbers, then

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

Oral Tell the simplest fraction for each difference below.

Written Copy. Find each difference in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{7}{8} - \frac{2}{8}$	$\frac{5}{6} - \frac{2}{6}$	$\frac{4}{5} - \frac{1}{5}$
2. $\frac{7}{10} - \frac{3}{10}$	$\frac{3}{4} - \frac{1}{4}$	$\frac{7}{7} - \frac{5}{7}$
3. $\frac{4}{3} - \frac{1}{3}$	$\frac{7}{8} - \frac{1}{8}$	$\frac{10}{12} - \frac{3}{12}$
4. $\frac{8}{12} - \frac{3}{12}$	$\frac{11}{16} - \frac{8}{16}$	$\frac{9}{11} - \frac{3}{11}$
5. $\frac{17}{20} - \frac{7}{20}$	$\frac{5}{16} - \frac{3}{16}$	$\frac{2}{9} - \frac{7}{9}$

What denominator would be most convenient to name each difference below as a single fraction?

<i>a</i>	<i>b</i>	<i>c</i>
6. $\frac{5}{12} - \frac{5}{24}$	$\frac{3}{4} - \frac{3}{8}$	$\frac{2}{3} - \frac{1}{5}$
7. $\frac{5}{8} - \frac{1}{3}$	$\frac{3}{7} - \frac{1}{3}$	$\frac{2}{3} - \frac{1}{6}$
8. $\frac{3}{10} - \frac{1}{4}$	$\frac{4}{5} - \frac{1}{2}$	$\frac{1}{2} - \frac{1}{3}$
9. $\frac{3}{4} - \frac{1}{2}$	$\frac{5}{6} - \frac{1}{2}$	$\frac{6}{7} - \frac{2}{3}$
10. $\frac{3}{4} - \frac{1}{3}$	$\frac{4}{5} - \frac{1}{3}$	$\frac{3}{4} - \frac{1}{10}$

Explain what is done to find the difference in simplest form in the example below.

<i>a</i>	<i>b</i>
11. $\frac{3}{5} - \frac{1}{3} = \frac{9}{15} - \frac{5}{15}$ $= \frac{9-5}{15}$ $= \frac{4}{15}$	$\frac{3}{5} \rightarrow \frac{9}{15}$ $-\frac{1}{3} \rightarrow -\frac{5}{15}$ $\hline \frac{4}{15}$
12. $\frac{11}{12} - \frac{1}{3} = \frac{11}{12} - \frac{4}{12}$ $= \frac{11-4}{12}$ $= \frac{7}{12}$	$\frac{11}{12} \rightarrow \frac{11}{12}$ $-\frac{1}{3} \rightarrow -\frac{4}{12}$ $\hline \frac{7}{12}$

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{7}{8} - \frac{1}{8}$	$\frac{9}{10} - \frac{7}{10}$	$\frac{17}{24} - \frac{9}{24}$
2. $\frac{9}{4} - \frac{7}{4}$	$\frac{7}{9} - \frac{0}{9}$	$\frac{11}{15} - \frac{7}{15}$
3. $\frac{5}{6} - \frac{1}{6}$	$\frac{1}{3} - \frac{1}{3}$	$\frac{9}{10} - \frac{5}{10}$
4. $\frac{7}{8} - \frac{3}{4}$	$\frac{2}{3} - \frac{1}{6}$	$\frac{2}{3} - \frac{2}{9}$
5. $\frac{3}{5} - \frac{1}{2}$	$\frac{4}{5} - \frac{1}{4}$	$\frac{3}{5} - \frac{7}{10}$
6. $\frac{7}{12} - \frac{3}{4}$	$\frac{1}{10} - \frac{3}{5}$	$\frac{3}{4} - \frac{5}{6}$
7. $\frac{7}{8}$ $-\frac{1}{6}$ \hline	$\frac{2}{3}$ $-\frac{1}{9}$ \hline	$\frac{3}{7}$ $-\frac{1}{4}$ \hline
8. $\frac{5}{6}$ $-\frac{1}{8}$ \hline	$\frac{5}{8}$ $-\frac{1}{4}$ \hline	$\frac{3}{4}$ $-\frac{2}{3}$ \hline
9. $\frac{7}{12}$ $-\frac{1}{6}$ \hline	$\frac{5}{6}$ $-\frac{1}{2}$ \hline	$\frac{7}{10}$ $-\frac{1}{3}$ \hline
10. $\frac{1}{4}$ $-\frac{1}{10}$ \hline	$\frac{5}{7}$ $-\frac{1}{2}$ \hline	$\frac{2}{2}$ $-\frac{2}{11}$ \hline

Can you do this? In each row, each number after the first differs from the preceding number by the same amount. Find this difference and name the next 5 numbers in each row.

- $\frac{1}{20}, \frac{3}{20}, \frac{1}{4}, \frac{7}{20}, \frac{9}{20}, \dots$
- $\frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \frac{1}{4}, \frac{5}{16}, \dots$
- $\frac{1}{24}, \frac{1}{6}, \frac{7}{24}, \frac{5}{12}, \frac{13}{24}, \dots$
- $(+\frac{7}{8}), (+\frac{1}{2}), (+\frac{1}{8}), (-\frac{1}{4}), \dots$

Mixed Numerals in Subtraction

To subtract a rational number from a whole number, compare the methods shown below.

$ \begin{aligned} 23 - \frac{5}{7} &= (22 + 1) - \frac{5}{7} \\ &= 22 + (1 - \frac{5}{7}) \\ &= 22 + (\frac{7}{7} - \frac{5}{7}) \\ &= 22 + \frac{2}{7} \\ &= 22\frac{2}{7} \end{aligned} $	$ \begin{array}{r} 23 \\ - \frac{5}{7} \\ \hline 22\frac{2}{7} \end{array} $	$ \begin{aligned} 23 - \frac{5}{7} &= \frac{161}{7} - \frac{5}{7} \\ &= \frac{161 - 5}{7} \\ &= \frac{156}{7} \\ &= 22\frac{2}{7} \end{aligned} $
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Which of the above methods do you consider most convenient? Why?

To subtract $4\frac{5}{9}$ from $10\frac{7}{9}$, compare the following methods.

$ \begin{array}{r} 10\frac{7}{9} \\ - 4\frac{5}{9} \\ \hline \end{array} $	$ \begin{aligned} 10 + \frac{7}{9} \\ - (4 + \frac{5}{9}) \\ \hline 6 + \frac{2}{9} = 6\frac{2}{9} \end{aligned} $	$ \begin{aligned} 10\frac{7}{9} - 4\frac{5}{9} &= (10 - 4) + (\frac{7}{9} - \frac{5}{9}) \\ &= 6 + \frac{2}{9} \\ &= 6\frac{2}{9} \end{aligned} $
--	---	---

How is 6 obtained in either method above? How is $\frac{2}{9}$ obtained? Explain how you can find the difference between $10\frac{7}{9}$ and $4\frac{5}{9}$.

Now let us subtract $1\frac{3}{4}$ from $9\frac{5}{6}$. Compare the methods below.

$ \begin{array}{r} 9\frac{5}{6} \\ - 1\frac{3}{4} \\ \hline \end{array} $	$ \begin{aligned} 9\frac{10}{12} \\ - 1\frac{9}{12} \\ \hline 8\frac{1}{12} \end{aligned} $	$ \begin{aligned} 9\frac{5}{6} - 1\frac{3}{4} &= (9 - 1) + (\frac{5}{6} - \frac{3}{4}) \\ &= 8 + \frac{10 - 9}{12} \\ &= 8 + \frac{1}{12} \text{ or } 8\frac{1}{12} \end{aligned} $
---	--	---

In either method, why are $\frac{5}{6}$ and $\frac{3}{4}$ renamed? Does $\frac{5}{6} - \frac{3}{4}$ name a positive number?

In the following example, the difference between the fractional numbers is not a positive number. Notice how the subtraction is completed in each example.

$ \begin{array}{r} 6\frac{1}{3} \\ - 1\frac{1}{2} \\ \hline \end{array} $	$ \begin{aligned} 6\frac{2}{6} \\ - 1\frac{3}{6} \\ \hline \end{array} $	$ \begin{aligned} 5\frac{8}{6} \\ - 1\frac{3}{6} \\ \hline 4\frac{5}{6} \end{aligned} $	$ \begin{aligned} 6\frac{1}{3} - 1\frac{1}{2} &= (6 - 1) + (\frac{1}{3} - \frac{1}{2}) \\ &= 5 + \frac{2 - 3}{6} \\ &= 5 + (-\frac{1}{6}) \text{ or } 5 - \frac{1}{6} \text{ or } 4\frac{5}{6} \end{aligned} $
---	--	--	--

Oral Tell which rational number is named by each expression below.

a	b	c
1. $7 - \frac{2}{3}$	$17 - \frac{3}{5}$	$136 - \frac{3}{4}$
2. $8 - \frac{11}{12}$	$31 - \frac{5}{9}$	$257 - \frac{6}{13}$

Tell how you would find the simplest numeral for each difference below.

a	b	c
3. $5\frac{5}{7} - 3\frac{1}{7}$	$3\frac{1}{2} - 1\frac{1}{3}$	$9\frac{1}{3} - 4\frac{2}{3}$
4. $8\frac{7}{8} - 2\frac{1}{2}$	$9\frac{3}{5} - 3\frac{1}{2}$	$17\frac{5}{8} - \frac{1}{6}$

Written Find the simplest numeral for each difference below by using the following method.

$$\begin{aligned}
 7\frac{4}{5} - 2\frac{1}{2} &= (7-2) + \left(\frac{4}{5} - \frac{1}{2}\right) \\
 &= 5 + \frac{8-5}{10} \\
 &= 5 + \frac{3}{10} \text{ or } 5\frac{3}{10}
 \end{aligned}$$

a	b	c
1. $3\frac{1}{2} - 1\frac{1}{3}$	$4\frac{2}{3} - \frac{1}{8}$	$4\frac{3}{4} - 1\frac{2}{3}$
2. $4\frac{7}{8} - 2\frac{3}{8}$	$5\frac{4}{5} - 2\frac{1}{4}$	$8\frac{5}{6} - 7\frac{3}{4}$
3. $7\frac{2}{3} - 2\frac{1}{2}$	$8\frac{3}{4} - 2\frac{1}{3}$	$5\frac{7}{8} - 3\frac{1}{2}$
4. $6\frac{2}{3} - 1\frac{1}{8}$	$9\frac{1}{2} - 2\frac{1}{6}$	$7\frac{3}{4} - 1\frac{3}{8}$
5. $9\frac{15}{16} - \frac{3}{4}$	$6\frac{11}{12} - \frac{2}{3}$	$11\frac{4}{9} - \frac{1}{6}$
6. $8\frac{3}{4} - 2\frac{2}{9}$	$11\frac{5}{6} - \frac{3}{8}$	$3\frac{3}{4} - 1\frac{1}{6}$
7. $6\frac{1}{6} - 3\frac{5}{6}$	$7\frac{3}{4} - 5\frac{11}{12}$	$9\frac{3}{5} - 4\frac{2}{3}$

Copy. Find each difference in simplest form.

a	b	c
8. $13\frac{1}{2} - 8\frac{4}{7}$	$34\frac{1}{3} - 22\frac{1}{15}$	$236\frac{3}{4} - 157\frac{1}{8}$
9. $4\frac{2}{3} - 2\frac{1}{6}$	$17\frac{1}{9} - 16\frac{2}{3}$	$98\frac{11}{18} - 64\frac{7}{9}$
10. $9\frac{3}{10} - 7\frac{2}{5}$	$47\frac{2}{3} - 39\frac{1}{6}$	$507\frac{3}{8} - 269\frac{5}{6}$
11. $7\frac{19}{20} - 5\frac{3}{4}$	$34\frac{2}{5} - 28\frac{7}{15}$	$375\frac{1}{4} - 257\frac{3}{8}$

Can you do this? Study the following subtraction.

$$\begin{aligned}
 1\frac{1}{2} - 2\frac{2}{3} &= (1-2) + \left(\frac{1}{2} - \frac{2}{3}\right) \\
 &= -1 + \frac{3-4}{6} \\
 &= -1 + \frac{-1}{6} \\
 &= -1\frac{1}{6}
 \end{aligned}$$

Use this method to find the simplest numeral for each expression below.

a	b	c
1. $6\frac{1}{3} - 9\frac{1}{2}$	$3\frac{1}{4} - 5\frac{2}{3}$	$2\frac{2}{5} - 5\frac{1}{2}$
2. $3\frac{1}{5} - 7\frac{1}{3}$	$4\frac{3}{5} - 7\frac{5}{8}$	$\frac{3}{10} - 17\frac{5}{6}$

Tell how Tell how you would complete the following subtraction.

$$\begin{aligned}
 2\frac{7}{8} - 7\frac{3}{8} &= (2-7) + \left(\frac{7}{8} - \frac{3}{8}\right) \\
 &= -5 + \frac{4}{8}
 \end{aligned}$$

Problem Solving

When $\frac{5}{6}$ of the members in a club are present, there are 4 members absent. How many members are there in the club?

To solve the problem, Sue reasoned as follows:

“I can think of the entire club as $\frac{6}{6}$. Then $\frac{6}{6} - \frac{5}{6}$ or $\frac{1}{6}$ of the members are absent. Since 4 members are absent, $\frac{1}{6}$ of the number of members in the club is 4. Let y represent the number of members in the club.”

$$\frac{1}{6}y = 4$$

$$y = 4 \div \frac{1}{6}$$

$$y = 4 \times 6$$

$$y = 24$$

Therefore, there are 24 members in the club.

Mr. Woods' car averages $17\frac{1}{2}$ miles per gallon of gasoline. How many gallons of gasoline did the car use on a trip of 147 miles?

To solve this problem, Tom reasoned as follows:

“I can assume that the car travels $17\frac{1}{2}$ miles on each gallon of gasoline. The number of miles per gallon times the number of gallons is equal to the number of miles driven. The number of miles per gallon is $17\frac{1}{2}$ and the number of miles driven is 147. Let n represent the number of gallons of gasoline.”

$$17\frac{1}{2}n = 147$$

$$n = 147 \div 17\frac{1}{2}$$

$$n = 8\frac{2}{5}$$

How can you find the simplest numeral for $147 \div 17\frac{1}{2}$? How many gallons of gasoline did the car use on this trip?

Oral Tell an open sentence that you can use to solve each problem below.

1. When $\frac{3}{8}$ of the pupils in a class were absent, 20 were present. How many pupils were enrolled in the class?

2. A car can travel $14\frac{5}{8}$ miles on a gallon of gasoline. How far can that car travel on 20 gallons of gasoline?

3. Jane worked $\frac{2}{5}$ of her math assignment during study period. She worked $\frac{1}{3}$ of the assignment after school. What part of the assignment remained to be done?

4. It took some scouts $1\frac{1}{2}$ hours to hike to a lake. There they fished for $1\frac{3}{4}$ hours. Then it took them $1\frac{2}{3}$ hours to make camp and prepare a meal. How much time did they spend in these activities?

5. Mrs. Boyd can save about $1\frac{1}{2}$ books of trading stamps in a month. How long will it take her to save 5 books of trading stamps?

6. Mrs. Holt purchased $1\frac{1}{2}$ pounds of ground meat, $2\frac{3}{4}$ pounds of steak, and a $3\frac{1}{2}$ pound roast. How many pounds of meat did she buy?

7. A boat traveled $50\frac{7}{10}$ miles in 3 hours. What was its speed in miles per hour?

8. A certain book weighs $2\frac{5}{8}$ pounds. If a stack of the books weighs $55\frac{1}{8}$ pounds, how many books are in the stack?

9. Max weighs $96\frac{1}{2}$ pounds and his dad weighs $186\frac{1}{4}$ pounds. What is their combined weight? What is the difference between their weights?

10. Molly ate $\frac{1}{4}$ of a bag of gumdrops. Later she ate $\frac{1}{4}$ of what was left. What part of the bag of gumdrops remained then?

11. The larger of two interlocking gears causes the smaller gear to revolve $3\frac{1}{4}$ times while it revolves once. How many times will the larger gear revolve if the smaller gear revolves 26 times?

Written Write an open sentence for each problem in *Oral*. Solve the open sentence. Then write an answer for each problem.

Can you do this? Two pipes, A and B, can be used to fill a swimming pool. Pipe A alone can fill the pool in 2 hours, which means that in 1 hour it can fill $\frac{1}{2}$ of the pool. Pipe B alone can fill the pool in 3 hours, which means that in 1 hour it can fill $\frac{1}{3}$ of the pool. If the pool is empty and both pipes are opened, what part of the pool can they fill in 1 hour? How long will it take both pipes to completely fill the pool?

Practice with Rational Numbers

Part 1 Write three equivalent fractions for each rational number listed below.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
1.	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{4}$
2.	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{10}$
3.	$\frac{5}{6}$	$\frac{1}{8}$	$\frac{3}{5}$	$\frac{2}{9}$	$\frac{5}{8}$

Part 2 Copy. Find each sum in simplest form.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\frac{3}{7} + \frac{2}{7}$	$\frac{1}{8} + \frac{6}{8}$	$\frac{5}{12} + \frac{7}{12}$
2.	$\frac{1}{3} + \frac{3}{4}$	$\frac{7}{8} + \frac{2}{3}$	$\frac{5}{8} + \frac{5}{6}$
3.	$4\frac{2}{3} + 1\frac{1}{2}$	$6\frac{1}{4} + 3\frac{1}{3}$	$7\frac{1}{8} + 4\frac{1}{2}$
4.	$\begin{array}{r} 9\frac{1}{5} \\ + 6\frac{3}{10} \\ \hline \end{array}$	$\begin{array}{r} 4\frac{1}{8} \\ + 6\frac{2}{3} \\ \hline \end{array}$	$\begin{array}{r} 17\frac{5}{6} \\ + 2\frac{1}{4} \\ \hline \end{array}$
5.	$\begin{array}{r} 1\frac{7}{8} \\ + 2\frac{5}{12} \\ \hline \end{array}$	$\begin{array}{r} 7\frac{2}{3} \\ + 5\frac{1}{4} \\ \hline \end{array}$	$\begin{array}{r} 1\frac{5}{6} \\ + 2\frac{1}{2} \\ \hline \end{array}$
6.	$\begin{array}{r} 17\frac{1}{4} \\ 16\frac{1}{8} \\ + 12\frac{3}{16} \\ \hline \end{array}$	$\begin{array}{r} 29\frac{7}{8} \\ 14\frac{1}{4} \\ + 42\frac{5}{12} \\ \hline \end{array}$	$\begin{array}{r} 41\frac{5}{6} \\ 16\frac{3}{4} \\ + 27\frac{2}{3} \\ \hline \end{array}$

Part 3 Copy. Replace each \bullet with either $<$ or $>$ to make each sentence become true.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\frac{1}{2} \bullet \frac{3}{8}$	$\frac{0}{12} \bullet \frac{1}{3}$	$\frac{1}{3} \bullet \frac{1}{4}$
2.	$\frac{6}{12} \bullet \frac{9}{19}$	$\frac{1}{2} \bullet \frac{5}{8}$	$\frac{4}{4} \bullet \frac{4}{5}$

3.	$\frac{2}{3} \bullet \frac{3}{4}$	$\frac{1}{4} \bullet \frac{1}{2}$	$\frac{1}{7} \bullet \frac{1}{8}$
4.	$\frac{1}{10} \bullet \frac{0}{5}$	$\frac{9}{16} \bullet \frac{7}{16}$	$\frac{3}{7} \bullet \frac{8}{9}$
5.	$\frac{7}{8} \bullet \frac{7}{9}$	$\frac{4}{3} \bullet \frac{3}{4}$	$\frac{7}{8} \bullet \frac{8}{7}$

Part 4 Copy. Find each difference in simplest form.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\frac{3}{4} - \frac{1}{2}$	$\frac{5}{6} - \frac{1}{2}$	$\frac{2}{3} - \frac{2}{5}$
2.	$4\frac{7}{8} - 2\frac{1}{4}$	$7\frac{5}{6} - 3\frac{1}{2}$	$4\frac{3}{4} - 1\frac{1}{12}$
3.	$7 - 1\frac{2}{3}$	$6 - 1\frac{1}{4}$	$9\frac{3}{5} - 7$
4.	$\begin{array}{r} 9\frac{2}{3} \\ - 5\frac{3}{4} \\ \hline \end{array}$	$\begin{array}{r} 17\frac{1}{2} \\ - 9\frac{5}{6} \\ \hline \end{array}$	$\begin{array}{r} 21\frac{1}{3} \\ - 8\frac{4}{5} \\ \hline \end{array}$

Part 5 Copy. Place one pair of () and one pair of [] in each sentence to make the sentence true.

- $\frac{1}{2} + \frac{2}{3} - \frac{3}{4} \div \frac{4}{5} = \frac{25}{48}$
- $\frac{1}{2} + \frac{3}{4} - \frac{2}{3} \div \frac{4}{5} = \frac{35}{48}$
- $\frac{1}{2} + \frac{3}{4} - \frac{2}{3} \div \frac{4}{5} = \frac{29}{48}$
- $\frac{1}{2} \times \frac{3}{4} + \frac{2}{3} \div \frac{5}{4} = \frac{5}{6}$

Can you do this? Recall that 4^2 means 4×4 . Similarly, $(\frac{5}{8})^2$ means $\frac{5}{8} \times \frac{5}{8}$. Tell the number named by each expression below.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
$(\frac{3}{4})^2$	$(-\frac{3}{4})^2$	$(\frac{2}{5})^2$	$(-\frac{2}{5})^2$

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Multiplying both the numerator and the denominator by the same number, other than zero, does not change the rational number. (133)

2. Let $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers.

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

If $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$. (134)

3. If two rational numbers have the same denominator, the one with the greater numerator is the greater number. (136)

4. If two rational numbers have the same numerator, the one with the greater denominator is the lesser number. (137)

5. Addition of rational numbers is commutative and associative, and zero is the identity number of addition of rational numbers. (144)

6. Subtracting a rational number is equivalent to adding its opposite. (146)

Words to Know

1. Equivalent fractions (134)

2. Least common multiple (138)

3. Least common denominator (140)

Questions to Discuss

1. How can you determine whether or not two fractions are equivalent? (134)

2. How can you determine which is greater: $\frac{5}{7}$ or $\frac{6}{7}$? $\frac{9}{13}$ or $\frac{10}{13}$? $\frac{7}{12}$ or $\frac{10}{13}$? (136)

3. How can you determine the least common multiple of 8, 12, and 18? The least common denominator of $\frac{5}{8}$, $\frac{7}{12}$, and $\frac{1}{18}$? (138, 140)

4. How would you find the sum of $\frac{7}{8}$ and $\frac{2}{5}$? (140)

5. How would you find the sum of $17\frac{5}{6}$ and $9\frac{1}{4}$? (140)

6. How would you find the difference between $\frac{5}{8}$ and $\frac{2}{5}$? Between $13\frac{2}{3}$ and $3\frac{4}{5}$? (146, 148)

Written Practice

Find the simplest numeral for each expression below.

a	b	
1. $\frac{3}{8} + \frac{1}{6}$	$\frac{2}{7} + \frac{4}{5}$	(141)

2. $\frac{1}{4} + \frac{5}{6} + \frac{2}{3}$	$2\frac{1}{2} + \frac{1}{5} + \frac{4}{7}$	(143)
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3. $\frac{7}{12} - \frac{3}{12}$	$\frac{4}{5} - \frac{2}{3}$	(147)
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4. $3\frac{4}{5} - \frac{7}{15}$	$8\frac{4}{9} - 3\frac{5}{6}$	(149)
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Self-Evaluation

Part 1 Copy. Find each sum in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{5}{8} + \frac{3}{4}$	$\frac{3}{16} + \frac{1}{2}$	$\frac{5}{6} + \frac{1}{3}$
2. $\frac{2}{3} + \frac{3}{4}$	$\frac{5}{6} + \frac{1}{2}$	$\frac{7}{8} + \frac{3}{4}$
3. $\frac{7}{12} + \frac{1}{4}$	$\frac{1}{2} + \frac{9}{10}$	$\frac{9}{10} + \frac{3}{5}$
4. $\frac{2}{5} + \frac{1}{2}$	$\frac{1}{2} + \frac{1}{6}$	$\frac{1}{2} + \frac{3}{4}$
5. $3\frac{1}{2} + 4\frac{2}{3} + 2\frac{1}{6}$	$3\frac{1}{3} + 6\frac{3}{4} + 2\frac{1}{2}$	$13\frac{1}{4} + 3\frac{1}{2} + 6\frac{3}{8}$
6. $4\frac{1}{3} + 2\frac{1}{2} + 6\frac{1}{8}$	$9\frac{1}{4} + 3\frac{1}{6} + 5\frac{7}{12}$	$11\frac{5}{16} + 2\frac{7}{8} + 3\frac{3}{4}$

Part 2 Copy. Find each difference in simplest form.


<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{3}{8} - \frac{3}{8}$	$\frac{9}{10} - \frac{7}{10}$	$\frac{19}{24} - \frac{7}{24}$
2. $\frac{2}{3} - \frac{1}{6}$	$\frac{4}{5} - \frac{1}{4}$	$\frac{4}{5} - \frac{1}{10}$
3. $\frac{7}{12} - \frac{1}{6}$	$\frac{5}{6} - \frac{1}{2}$	$\frac{6}{7} - \frac{1}{2}$
4. $7 - \frac{2}{3}$	$8 - 1\frac{2}{3}$	$5 - 2\frac{3}{8}$
5. $3\frac{3}{4} - 1\frac{1}{4}$	$4\frac{5}{6} - 1\frac{1}{6}$	$8\frac{7}{9} - 3\frac{5}{9}$
6. $3\frac{1}{6} - 1\frac{1}{2}$	$9\frac{3}{8} - 2\frac{3}{4}$	$11\frac{1}{4} - 5\frac{1}{2}$

Part 3 Copy. Place one pair of () and one pair of [] so that each sentence becomes true.

- $\frac{1}{2} + \frac{1}{3} \times \frac{1}{4} \div \frac{1}{5} = \frac{11}{12}$
- $\frac{1}{2} + \frac{1}{3} \times \frac{1}{4} \div \frac{1}{5} = 1\frac{1}{24}$
- $\frac{1}{2} + \frac{1}{3} \times \frac{1}{4} \div \frac{1}{5} = 2\frac{11}{12}$
- $\frac{7}{8} - \frac{1}{2} \times \frac{1}{4} \div \frac{2}{3} = 1\frac{1}{8}$
- $\frac{7}{8} - \frac{1}{2} \times \frac{1}{4} \div \frac{2}{3} = \frac{9}{64}$

Part 4 Copy. Show that the fractions in each pair name the same number.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{2}{3}, \frac{4}{6}$	$\frac{6}{8}, \frac{3}{4}$	$\frac{6}{12}, \frac{3}{6}$
2. $\frac{8}{12}, \frac{2}{3}$	$\frac{3}{4}, \frac{9}{12}$	$\frac{8}{10}, \frac{4}{5}$
3. $\frac{1}{5}, \frac{3}{15}$	$\frac{1}{2}, \frac{7}{14}$	$\frac{14}{16}, \frac{7}{8}$
4. $\frac{5}{6}, \frac{10}{12}$	$\frac{1}{4}, \frac{2}{8}$	$\frac{1}{4}, \frac{3}{12}$

Part 5 Copy. Replace  with either > or < so as to make each sentence become true.

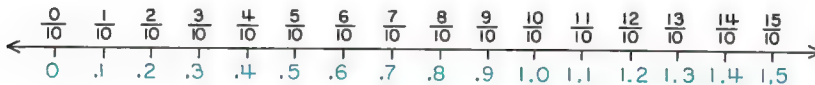
<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{1}{2} \text{ } \frac{2}{3}$	$\frac{3}{4} \text{ } \frac{5}{6}$	$\frac{7}{16} \text{ } \frac{5}{12}$
2. $\frac{5}{6} \text{ } \frac{3}{4}$	$\frac{3}{7} \text{ } \frac{3}{6}$	$\frac{9}{10} \text{ } \frac{8}{9}$
3. $\frac{7}{8} \text{ } \frac{7}{12}$	$\frac{5}{16} \text{ } \frac{5}{12}$	$\frac{5}{8} \text{ } \frac{3}{8}$
4. $\frac{17}{24} \text{ } \frac{5}{12}$	$\frac{8}{5} \text{ } \frac{4}{7}$	$\frac{1}{3} \text{ } \frac{4}{11}$

Chapter 7

DECIMALS

Decimal Numerals

Rational numbers like $\frac{7}{10}$, $\frac{13}{100}$, and $\frac{29}{1000}$ have powers of ten as denominators. You can devise a way to name such rational numbers without having to write or print fractions.



Each numeral below the number line names the same number as the fraction directly above it. Does the numeral in .3 name the numerator or the denominator of $\frac{3}{10}$? How does the number of 0's in 10 compare with the number of digits to the right of the . in .3? Would your answers to the above questions be the same for other numerals like .5 or 1.3?

Since $\frac{13}{10} = \frac{10+3}{10} = \frac{10}{10} + \frac{3}{10} = 1 + \frac{3}{10}$ or $1\frac{3}{10}$, notice that 1.3 corresponds to $1\frac{3}{10}$ and also to $\frac{13}{10}$.

Numerals like .3 or 1.3 are called **decimal numerals** or simply *decimals*. You read .3 and 1.3 just as you read their corresponding fractions or mixed numerals.

.3 is read *three tenths*.

1.3 is read *one and three tenths*.

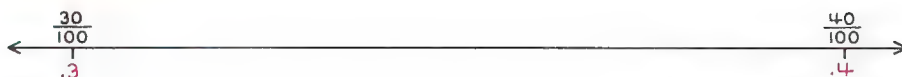
Oral Read each decimal below.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	.7	.2	1.4	5.3
2.	.9	.5	1.8	6.7

Written Do the following.

1-2. Write a fraction or a mixed numeral for each decimal in *Oral*. Be sure each fraction or mixed numeral is in simplest form.

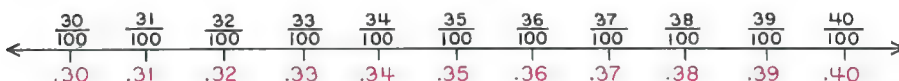
Decimal Numerals



This number line provides a more detailed view of the section from .3 to .4. You already know there are many names for every rational number. For example,

$$.3 = \frac{3}{10} = \frac{30}{100} = \frac{300}{1000}, \text{ and so on.}$$

Why is it correct to use $\frac{30}{100}$ for .3? To use $\frac{40}{100}$ for .4? How many hundredths are there between $\frac{30}{100}$ and $\frac{40}{100}$?



Does the numeral in .33 name the numerator or the denominator of $\frac{33}{100}$? How does the number of 0's in 100 compare with the number of digits to the right of the decimal point in .33? Would your answers to these questions be the same for other numerals like .35 or .38?

It is easy to see how decimal digits obtain their meaning.

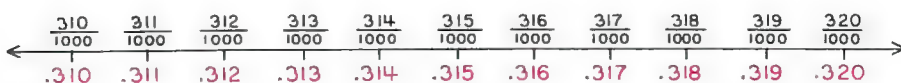
$$\frac{37}{100} = \frac{30+7}{100} = \frac{30}{100} + \frac{7}{100} = \frac{3}{10} + \frac{7}{100}$$

$$\text{Therefore, } .37 = \frac{3}{10} + \frac{7}{100}.$$

Similarly, $\frac{7}{100}$ can be thought of as $\frac{0}{10} + \frac{7}{100}$, and hence as .07.

Since $\frac{31}{100} = \frac{31}{100} \times 1 = \frac{31}{100} \times \frac{10}{10} = \frac{310}{1000}$, you can also express .31 as $\frac{310}{1000}$. Why can you express .32 as $\frac{320}{1000}$?

Now let us take a more detailed look at the section of a number line from $\frac{310}{1000}$ to $\frac{320}{1000}$.



$$\frac{317}{1000} = \frac{300}{1000} + \frac{10}{1000} + \frac{7}{1000} = \frac{3}{10} + \frac{1}{100} + \frac{7}{1000}$$

$$\text{Therefore, } .317 = \frac{3}{10} + \frac{1}{100} + \frac{7}{1000}.$$

By following the pattern of decimal numerals for tenths, hundredths, and thousandths, it is possible to extend base-ten numeration to the right of the decimal point.

1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
Th	H	T	0	Ts	Hs	Ths
	4	7	2	0	8	5

four hundred seventy-two and eighty-five thousandths

This extended numeration pattern will eventually enable you to name any rational number by a decimal. As yet, it enables you to name only some rational numbers by decimals. Which ones?

Oral Read each decimal below.

<i>a</i>	<i>b</i>	<i>c</i>
1. .13	7.11	34.27
2. .71	5.09	153.01
3. .351	1.207	26.719
4. .003	6.043	126.059

Answer the following questions.

5. What would be the denominator corresponding to .27? To .07? To .127? To .027? To .1357? To .0017?

6. How many digits to the right of the decimal point in the decimal numeral correspond to a denominator of 100? Of 1000?

Written Do the following.

1-4. Write a fraction or a mixed numeral for each decimal in *Oral* 1-4.

For each fraction or mixed numeral below, write an equivalent decimal numeral.

<i>a</i>	<i>b</i>	<i>c</i>
5. $\frac{7}{10}$	$\frac{45}{100}$	$\frac{125}{1000}$
6. $\frac{3}{10}$	$\frac{9}{100}$	$\frac{413}{1000}$
7. $4\frac{9}{10}$	$5\frac{17}{100}$	$9\frac{357}{1000}$

Can you do this? Think of a rational number whose numerator is 1 and whose denominator is not a power of ten. Can it be named by a decimal by what we have presented thus far?

Changing Decimals to Fractions

By merely reading a decimal, you can easily change it to an equivalent fraction; that is, to a fraction that names the same number.

.4	<i>four tenths</i>	$\frac{4}{10}$
.27	<i>twenty-seven hundredths</i>	$\frac{27}{100}$
.049	<i>forty-nine thousandths</i>	$\frac{49}{1000}$

Notice that this method of changing a decimal to a fraction always yields a fraction whose denominator is a power of ten. However, the resulting fraction is not necessarily in simplest form. Which of the fractions above is not in simplest form? How can it be changed to simplest form?

Unless stated otherwise, when changing a decimal to an equivalent fraction, the fraction is to be in simplest form.

$$.4 = \frac{4}{10} = \frac{2}{5} \quad .40 = \frac{40}{100} = \frac{2}{5} \quad .400 = \frac{400}{1000} = \frac{2}{5}$$

How is $\frac{4}{10}$ changed to $\frac{2}{5}$? How is $\frac{40}{100}$ changed to $\frac{2}{5}$? How is $\frac{400}{1000}$ changed to $\frac{2}{5}$? Do .4, .40, and .400 all name the same number?

The pattern of changing a decimal to an equivalent fraction can be extended to numbers greater than 1.

$$5.17 = 5 + .17 = 5 + \frac{17}{100} = 5\frac{17}{100}$$
$$58.125 = 58 + .125 = 58 + \frac{125}{1000} = 58 + \frac{1}{8} = 58\frac{1}{8}$$

Do .3 and .30 name the same number? Explain your answer. Do 0.3 and .3 name the same number? What purpose might the 0 in 0.3 serve?

Do .3 and .03 name the same number? Explain your answer.

Oral When answering questions 1–2 below, think of the fraction that results from reading the decimal.

1. When changing .21 to an equivalent fraction, what will the numerator be? What will the denominator be? Will this fraction be in simplest form? If not, explain how to change it to simplest form.

2. When changing .012 to an equivalent fraction, what will the numerator be? What will the denominator be? Will this fraction be in simplest form? If not, explain how to change it to simplest form.

Answer the questions below.

3. How can you show that .7, .70, and .700 all name the same number?

4. How can you show that 0.4 and .4 name the same number?

Tell how you would express each decimal below as an equivalent fraction or mixed numeral.

	<i>a</i>	<i>b</i>	<i>c</i>
5.	.27	.41	.013
6.	.2	.48	.250
7.	3.2	4.25	1.375
8.	2.01	7.4	17.625
9.	.444	.101	39.275

Written Change each decimal below to an equivalent fraction in simplest form.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	.7	.07	.007	.700
2.	.70	.070	.0070	.0007
3.	.3	.30	.03	.030
4.	.50	.05	.050	.005
5.	.1	.01	.001	.0010
6.	.8	.80	.080	.0080
7.	.9	.09	.009	.090
8.	.6	.06	.600	.006
9.	.4	.54	.78	.482
10.	.5	.24	.08	.625

Another way To change a decimal to an equivalent fraction you can think as follows.

$$.37 = \frac{37}{10^2} = \frac{37}{100} \qquad .013 = \frac{13}{10^3} = \frac{13}{1000}$$

Is the number of digits to the right of the decimal point also the exponent of 10 in the denominator?

Use the above method to change the decimal numerals in *Oral* 5–6 to equivalent fractions in simplest form.

Changing Fractions to Decimals

If the denominator is a power of ten, it is easy to change a fraction to an equivalent decimal.

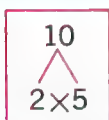
$$\frac{3}{10} = .3$$

$$\frac{17}{100} = .17$$

$$\frac{31}{1000} = .031$$

Suppose the denominator is not a power of ten. Then two possibilities arise: (1) the denominator is a factor of some power of ten or (2) the denominator is not a factor of any power of ten. Let us investigate the first possibility.

Let us restrict our discussion to fractions in simplest form. If other fractions occur, change them to simplest form first.



What are the prime factors of ten? Consider changing $\frac{1}{2}$ to an equivalent decimal. Then

$$\frac{1}{2} = \frac{a}{10}$$

where a names the number of *tenths*. Recall the relationship between equivalent fractions.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

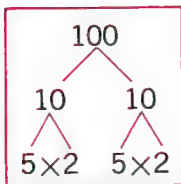
$$\frac{1}{2} = \frac{a}{10} \text{ so } 2a = 10 \text{ and } a = \frac{10}{2} \text{ or } 5.$$

$$\text{Therefore, } \frac{1}{2} = \frac{5}{10} \text{ or } .5.$$

$\frac{2}{5}$ can be changed to an equivalent decimal as follows.

$$\frac{2}{5} = \frac{x}{10} \text{ so } 5x = 20 \text{ and } x = \frac{20}{5} \text{ or } 4.$$

$$\text{Therefore, } \frac{2}{5} = \frac{4}{10} \text{ or } .4.$$



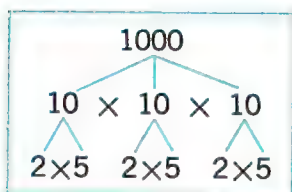
What are the prime factors of 100? Are these the same prime factors as for 10? In order that a fraction can be expressed as *hundredths* in decimal form, the possible denominators are

2, 4, 5, 10, 20, 25, 50, 100.

You already know how to change a fraction to an equivalent decimal if the denominator is 2, 5, 10, or 100. Hence, you need only consider denominators of 4, 20, 25, and 50. Consider $\frac{51}{75}$. Note that it is not in simplest form.

$$\frac{51}{75} = \frac{17}{25} \quad \frac{17}{25} = \frac{x}{100} \text{ so } 25x = 17 \times 100 \text{ and } x = \frac{17 \times 100}{25} \text{ or } 68.$$

Therefore, $\frac{17}{25} = \frac{68}{100}$ or .68.



Are the prime factors of 1000 the same as the prime factors of 10 and of 100? What are the only denominators you need to consider when changing a fraction to thousandths in decimal form? Consider $\frac{9}{40}$.

$$\frac{9}{40} = \frac{n}{1000} \text{ so } 40n = 9 \times 1000 \text{ and } n = \frac{9 \times 1000}{40} \text{ or } 225.$$

Therefore, $\frac{9}{40} = \frac{225}{1000}$ or .225.

Oral If the letters a, b, c , and so on represent whole numbers, which of the open sentences below have a whole number solution?

- | a | b | c |
|-----------------------------------|---------------------------------|-----------------------------------|
| 1. $\frac{3}{20} = \frac{a}{100}$ | $\frac{1}{8} = \frac{b}{100}$ | $\frac{47}{125} = \frac{c}{1000}$ |
| 2. $\frac{3}{40} = \frac{d}{100}$ | $\frac{3}{25} = \frac{e}{100}$ | $\frac{37}{50} = \frac{f}{100}$ |
| 3. $\frac{7}{80} = \frac{g}{100}$ | $\frac{7}{80} = \frac{h}{1000}$ | $\frac{3}{4} = \frac{i}{10}$ |

Answer the question below.

4. Is $\frac{9}{12}$ in simplest form? How would you express $\frac{9}{12}$ before changing it to an equivalent decimal?

Written Change each fraction below to an equivalent decimal having the least possible number of digits.

- | a | b | c | d | e |
|-------------------|-----------------|-----------------|------------------|------------------|
| 1. $\frac{3}{5}$ | $\frac{4}{25}$ | $\frac{23}{40}$ | $\frac{7}{28}$ | $\frac{12}{125}$ |
| 2. $\frac{3}{50}$ | $\frac{5}{8}$ | $\frac{3}{75}$ | $\frac{21}{60}$ | $\frac{29}{500}$ |
| 3. $\frac{2}{4}$ | $\frac{16}{20}$ | $\frac{12}{16}$ | $\frac{12}{375}$ | $\frac{29}{100}$ |

Can you do this? Show that $\frac{3}{5}$, .6, .60, and .600 all name the same number.

Tell how How would you change $\frac{11}{625}$ to an equivalent decimal?

Changing Fractions to Decimals

You have already learned how to change a fraction to an equivalent decimal when the denominator is a factor of a power of ten. Such denominators have prime factors of only 2, only 5, or both 2 and 5. The resulting decimals always name a whole number of *tenths*, *hundredths*, *thousandths*, and so on. Such decimals are called **terminating decimals**.

Now let us consider denominators that contain at least one prime factor other than 2 or 5. Consider changing $\frac{1}{3}$ to an equivalent decimal.

$$\begin{array}{rcl} \frac{1}{3} = \frac{n}{10} & \frac{1}{3} = \frac{n}{100} & \frac{1}{3} = \frac{n}{1000} \\ 3n = 10 & 3n = 100 & 3n = 1000 \\ n = \frac{10}{3} & n = \frac{100}{3} & n = \frac{1000}{3} \end{array}$$

Is 3 a factor of 10? Of 100? Of 1000? Of any power of 10? Is a whole number named by $\frac{10}{3}$? By $\frac{100}{3}$? By $\frac{1000}{3}$? Do you think $\frac{1}{3}$ can ever be named by a terminating decimal?

Consider changing $\frac{5}{11}$ to an equivalent decimal.

$$\begin{array}{rcl} \frac{5}{11} = \frac{m}{10} & \frac{5}{11} = \frac{m}{100} & \frac{5}{11} = \frac{m}{1000} \\ 11m = 50 & 11m = 500 & 11m = 5000 \\ m = \frac{50}{11} & m = \frac{500}{11} & m = \frac{5000}{11} \end{array}$$

Is a whole number named by $\frac{50}{11}$? By $\frac{500}{11}$? By $\frac{5000}{11}$? Do you think $\frac{5}{11}$ can ever be named by a terminating decimal?

If the denominator has a prime factor other than 2 or 5, then the fraction cannot be changed to a terminating decimal.

In a later lesson we will discuss how you can determine decimal approximations for such fractions as $\frac{1}{3}$ and $\frac{5}{11}$. For now it is important to recognize only whether or not a fraction can be expressed as a terminating decimal.

Consider the fraction $\frac{12}{80}$. Since it is not in simplest form, change it to simplest form first.

$$\frac{12}{80} = \frac{3 \times 4}{20 \times 4} = \frac{3}{20} \times \frac{4}{4} = \frac{3}{20} \times 1 = \frac{3}{20}$$

Since $20 = 2 \times 2 \times 5$, its only prime factors are 2 and 5. Hence, $\frac{12}{80}$ or $\frac{3}{20}$ can be expressed as a terminating decimal.

Consider the fraction $\frac{4}{15}$. It is already in simplest form. What are the prime factors of 15? Does 15 have at least one prime factor other than 2 or 5? If so, $\frac{4}{15}$ cannot be expressed as a terminating decimal.

Oral The prime factors are given for each denominator below. Tell whether or not each fraction can be changed to a terminating decimal.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\frac{3}{16}$ $2 \times 2 \times 2 \times 2$	$\frac{7}{15}$ 3×5	$\frac{6}{13}$ 13
2.	$\frac{7}{40}$ $2 \times 2 \times 2 \times 5$	$\frac{8}{55}$ 5×11	$\frac{19}{50}$ $2 \times 5 \times 5$

Each fraction below is in simplest form. Tell the prime factors of the denominator for each fraction. Then determine whether or not the fraction can be changed to a terminating decimal.

	<i>a</i>	<i>b</i>	<i>c</i>
3.	$\frac{5}{8}$	$\frac{3}{14}$	$\frac{9}{20}$
4.	$\frac{24}{25}$	$\frac{14}{75}$	$\frac{26}{125}$
5.	$\frac{7}{9}$	$\frac{19}{30}$	$\frac{9}{28}$

Written Change each fraction below to simplest form and then to an equivalent decimal.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$\frac{21}{28}$	$\frac{15}{40}$	$\frac{12}{75}$	$\frac{39}{65}$
2.	$\frac{21}{150}$	$\frac{7}{35}$	$\frac{35}{125}$	$\frac{12}{64}$

Another way To change $\frac{9}{40}$ to an equivalent decimal, first express the denominator as a product of primes.

$$\frac{9}{40} = \frac{9}{2 \times 2 \times 2 \times 5}$$

If there were two more factors of 5 in the denominator, each 2 could be paired with a 5 and the denominator would be a power of 10. Multiply by 1 in the form $\frac{5 \times 5}{5 \times 5}$.

$$\begin{aligned} \frac{9}{2 \times 2 \times 2 \times 5} \times \frac{5 \times 5}{5 \times 5} &= \frac{9 \times 5 \times 5}{2 \times 2 \times 2 \times 5 \times 5 \times 5} \\ &= \frac{225}{10 \times 10 \times 10} \text{ or } \frac{225}{1000} \end{aligned}$$

Therefore, $\frac{9}{40} = .225$.

Use this method to do *Written* 1.

Addition with Decimals

Since some rational numbers can be named by either terminating decimals or fractions, you have a choice of forms when adding such rational numbers. In some cases one form is more convenient than the other. Which form do you think is more convenient in each example below?

A $\frac{1}{5} + \frac{3}{5} = \frac{4}{5}$

$$\begin{array}{r} .2 \\ +.6 \\ \hline .8 \end{array}$$

B $\frac{3}{4} + \frac{4}{5} = \frac{15}{20} + \frac{16}{20} = \frac{31}{20} = 1\frac{11}{20}$

$$\begin{array}{r} .75 \\ +.80 \\ \hline 1.55 \end{array}$$

In example **B** above, $\frac{4}{5}$ can be changed to .8 or .80 or .800, and so on. Is it necessary to write the 0 of .80 in the example above?

To add 2.3 and 17.004, either method below is correct.

$$\begin{array}{r} 17.004 \\ +2.300 \\ \hline 19.304 \end{array}$$

$$\begin{array}{r} 17.004 \\ +2.3 \\ \hline 19.304 \end{array}$$

How can you show that $2.3 = 2.300$?

The decimal form enables you to add rational numbers by using the procedures for adding whole numbers. This is illustrated in the following example.

$$\begin{array}{l} 14\frac{13}{1000} = 14 + \frac{0}{10} + \frac{1}{100} + \frac{3}{1000} \\ 24\frac{202}{1000} = 24 + \frac{2}{10} + \frac{0}{100} + \frac{2}{1000} \\ + 3\frac{22}{1000} = 3 + \frac{0}{10} + \frac{2}{100} + \frac{2}{1000} \\ \hline 41\frac{237}{1000} = 41 + \frac{2}{10} + \frac{3}{100} + \frac{7}{1000} \end{array}$$

T	O	Ts	Hs	Ths	
1	4	0	1	3	14.013
2	4	2	0	2	24.202
+	3	0	2	2	+3.022
4	1	2	3	7	41.237

└─ Add the Ths. ─┐

└─ Add the Hs. ─┐

└─ Add the Ts. ─┐

└─ Add the Ones. ─┐

└─ Add the Tens. ─┐

Does aligning the decimal points vertically automatically place each digit in its correct place-value position?

In addition with decimals it is often necessary to rename the sum in one or more place-value positions. In which place-value positions below is it necessary to rename the sum?

$$\begin{array}{r} \overset{1}{21.3}\overset{1}{85} \\ +17.83 \\ \hline 39.215 \end{array} \quad \left\{ \begin{array}{l} \frac{8}{100} + \frac{3}{100} = \frac{8+3}{100} = \frac{11}{100} = \frac{10}{100} + \frac{1}{100} = \frac{1}{10} + \frac{1}{100} \\ \text{How is } \frac{1}{10} \text{ recorded? How is } \frac{1}{100} \text{ recorded?} \end{array} \right.$$

Explain how the sum $\frac{1}{10} + \frac{3}{10} + \frac{8}{10}$ is renamed and recorded in the above example.

Oral Each addition numeral below is given in fraction and in decimal form. Which form would you prefer in computing the sum?

1. $\frac{5}{8} + \frac{4}{25}$ $.625 + .16$

2. $\frac{3}{8} + \frac{2}{8}$ $.375 + .25$

Explain how the digits in the numeral for the sum are found in each example below.

<i>a</i>	<i>b</i>
<p>3. $\begin{array}{r} 7.004 \\ 5.34 \\ +4.8 \\ \hline 17.144 \end{array}$</p>	<p>$\begin{array}{r} 23.276 \\ 3.94 \\ +.015 \\ \hline 27.231 \end{array}$</p>

Written Change each addition numeral below to column algorithm form and compute each sum.

1. $2.4 + 6.82 + 7.019$

2. $6.04 + 2.703 + .8$

3. $2.094 + .24 + 5.675$

4. $17.28 + .0361 + 9.809$

Copy. Find each sum.

	<i>a</i>	<i>b</i>	<i>c</i>
5.	$\begin{array}{r} 71.9 \\ +32.1 \\ \hline \end{array}$	$\begin{array}{r} 8.4 \\ +.762 \\ \hline \end{array}$	$\begin{array}{r} 19.082 \\ +5.004 \\ \hline \end{array}$
6.	$\begin{array}{r} 17.09 \\ +3.413 \\ \hline \end{array}$	$\begin{array}{r} 18.049 \\ +2.4 \\ \hline \end{array}$	$\begin{array}{r} 7.005 \\ +3.98 \\ \hline \end{array}$
7.	$\begin{array}{r} 1.009 \\ .87 \\ +6.293 \\ \hline \end{array}$	$\begin{array}{r} 24.109 \\ 4.08 \\ +17.5 \\ \hline \end{array}$	$\begin{array}{r} 17.091 \\ 14.014 \\ +6.8 \\ \hline \end{array}$
8.	$\begin{array}{r} 7.842 \\ .309 \\ .7 \\ +4.009 \\ \hline \end{array}$	$\begin{array}{r} 125.92 \\ 14.98 \\ 310.5 \\ +6.07 \\ \hline \end{array}$	$\begin{array}{r} 12.809 \\ .73 \\ 78.41 \\ +6.007 \\ \hline \end{array}$

Write an open sentence for each problem below. Solve the open sentence. Answer the problem.

9. A chemist mixed 4.75 grams of one compound with 1.62 grams of another compound. How many grams were there in the mixture?

10. What is the total cost of these items: clock \$3.49, film \$2.56, drugs \$5.70, and cosmetics \$4.85?

Subtraction with Decimals

A	B	C	D
$\begin{array}{r} 5 \frac{729}{1000} \\ - 3 \frac{415}{1000} \\ \hline \end{array}$	$\begin{array}{r} 5 + \frac{7}{10} + \frac{2}{100} + \frac{9}{1000} \\ - (3 + \frac{4}{10} + \frac{1}{100} + \frac{5}{1000}) \\ \hline 2 + \frac{3}{10} + \frac{1}{100} + \frac{4}{1000} \end{array}$	$\begin{array}{r} \text{O} \quad \text{Ts} \text{Hs} \text{Ths} \\ 5.729 \\ - 3.415 \\ \hline 2.314 \end{array}$	$\begin{array}{r} 5.729 \\ - 3.415 \\ \hline 2.314 \end{array}$

How is the example in **B** different from that in **C**? What do you observe about the placement of the decimal points in **C**?

The procedures for subtraction of whole numbers can be used for subtraction with decimals provided the decimals are arranged so that all decimal points lie on the same vertical line.

If you cannot subtract in every place-value position, rename the minuend as you do for whole numbers. You can use fractions to verify that this renaming procedure holds for decimals.

$$\begin{array}{r} .83 \\ - .46 \\ \hline .37 \end{array} \quad \begin{array}{r} \text{Ts} | \text{Hs} \\ 7 | 13 \\ \cancel{8} \cancel{3} \\ - .46 \\ \hline .37 \end{array}$$

Consider finding the simplest numeral for $1 - .1$, for $1 - .01$, and for $1 - .001$. Tell how the two sentences in each row below are alike.

$$\begin{array}{ll} 10 - 1 = 9 & 1 - .1 = .9 \\ 100 - 1 = 99 & 1 - .01 = .99 \\ 1000 - 1 = 999 & 1 - .001 = .999 \end{array}$$

Recall that there are many names for every number.

$$1 = \frac{10}{10} = \frac{100}{100} = \frac{1000}{1000} = \dots \quad 1 = 1.0 = 1.00 = 1.000 = \dots$$

Explain each example below.

$$\begin{array}{r} 1.0 \\ - .1 \\ \hline .9 \end{array} \quad \begin{array}{r} 1.00 \\ - .01 \\ \hline .99 \end{array} \quad \begin{array}{r} 1.000 \\ - .001 \\ \hline .999 \end{array}$$

Oral Study the following example. Then answer questions 1-4 below.

$$\begin{array}{r|l}
 T & 0 \quad Ts \quad Hs \quad Ths \\
 & 2 \quad 15 \quad 16 \\
 3 & 3.665 \\
 -1 & 2.98 \\
 \hline
 2 & 0.685
 \end{array}$$

1. Why is the minuend renamed? How is it renamed?

2. How is each digit in the numeral for the difference obtained?

3. In subtraction with decimals why should you arrange the numerals so that all decimal points lie on the same vertical line?

4. How is subtraction with decimals similar to subtraction of whole numbers? How does it differ?

Answer the following questions.

5. To subtract 8.049 from 10.376, is it necessary to rename 10.376? Why or why not?

6. To subtract .2 from 3, how would you rename 3?

7. To subtract .54 from 17, how would you rename 17?

Tell the simplest numeral for each difference below.

	<i>a</i>	<i>b</i>	<i>c</i>
8.	$\begin{array}{r} 8.75 \\ -1.31 \\ \hline \end{array}$	$\begin{array}{r} 24.94 \\ -4.8 \\ \hline \end{array}$	$\begin{array}{r} 9.017 \\ -6.002 \\ \hline \end{array}$

Written Change each subtraction numeral below to column form. Then find the simplest numeral for the difference.

<i>a</i>	<i>b</i>
1. $20-.2$	$12.976-4.331$
2. $9-.015$	$2.782-.93$
3. $4-.54$	$29.31-17.642$

Copy. Find the simplest numeral for each difference.

	<i>a</i>	<i>b</i>	<i>c</i>
4.	$\begin{array}{r} 84.19 \\ -23.08 \\ \hline \end{array}$	$\begin{array}{r} 6.042 \\ -3.012 \\ \hline \end{array}$	$\begin{array}{r} 14.1407 \\ -6.0103 \\ \hline \end{array}$
5.	$\begin{array}{r} 52.06 \\ -21.83 \\ \hline \end{array}$	$\begin{array}{r} 8.407 \\ -3.194 \\ \hline \end{array}$	$\begin{array}{r} 42.4190 \\ -9.2488 \\ \hline \end{array}$
6.	$\begin{array}{r} 14.95 \\ -8.09 \\ \hline \end{array}$	$\begin{array}{r} 2.710 \\ -.806 \\ \hline \end{array}$	$\begin{array}{r} 31.0108 \\ -24.4289 \\ \hline \end{array}$
7.	$\begin{array}{r} 19 \\ -7.25 \\ \hline \end{array}$	$\begin{array}{r} 34.4 \\ -8.975 \\ \hline \end{array}$	$\begin{array}{r} 8.2 \\ -5.007 \\ \hline \end{array}$

Solve each problem below.

8. The diameter of the earth at the equator is 7926.68 miles and through the poles it is 7899.98 miles. How much greater is the diameter at the equator than through the poles?

9. The rainfall for May was 3.45 inches and for June it was 4.10 inches. By how many inches did the rainfall for June exceed the rainfall for May?

Multiplication with Decimals

$$.3 \times .25 = \frac{3}{10} \times \frac{25}{100} = \frac{3}{10} \times \frac{1}{4} = \frac{3 \times 1}{10 \times 4} = \frac{3}{40}$$

To multiply .25 by .3, you can change to fractional form as shown above. Without changing $\frac{25}{100}$ to simplest form, you can discover a method of multiplication with decimals.

$$.3 \times .25 = \frac{3}{10} \times \frac{25}{100} = \frac{3 \times 25}{10 \times 100} = \frac{75}{1000} = .075$$

It appears that the procedures for multiplying whole numbers can be used in multiplication with decimals. To find the simplest numeral for $.3 \times .25$, multiply 25 by 3 and then place the decimal point in the product numeral.

Write

$$.3 \times .25 \longrightarrow \begin{array}{r} 25 \\ \times 3 \\ \hline 75 \end{array} \text{ and } \frac{1}{10} \times \frac{1}{100} = \frac{1}{1000} \longrightarrow \begin{array}{r} .25 \\ \times .3 \\ \hline .075 \end{array}$$

Study the following sentences to discover a method for placing the decimal point in a product numeral.

3×4	$=$	3×4	$= 12$	$= 12$
$.3 \times 4$	$=$	$\frac{3}{10} \times 4$	$= \frac{12}{10}$	$= 1.2$
$3 \times .4$	$=$	$3 \times \frac{4}{10}$	$= \frac{12}{10}$	$= 1.2$
$.3 \times .4$	$=$	$\frac{3}{10} \times \frac{4}{10}$	$= \frac{12}{100}$	$= .12$
$.3 \times .04$	$=$	$\frac{3}{10} \times \frac{4}{100}$	$= \frac{12}{1000}$	$= .012$
$.03 \times .04$	$=$	$\frac{3}{100} \times \frac{4}{100}$	$= \frac{12}{10000}$	$= .0012$

Compare the number of digits after the decimal point in each product numeral above with the total number of digits after the decimal points in both decimal numerals to its left. What pattern can you discover?

Oral Use the following example to answer questions 1–3.

$$\begin{array}{r} 3.12 \longrightarrow 2 \text{ digits} \\ \times .4 \longrightarrow 1 \text{ digit} \\ \hline 1.248 \longleftarrow 3 \text{ digits} \end{array}$$

1. How are the digits in 1.248 obtained?

2. How can you determine where to place the decimal point in 1.248?

3. How is multiplication with decimals similar to multiplication of whole numbers?

You know that $5 \times 7 = 35$. State each product below in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
4. $5 \times .7$	$5 \times .07$	$.05 \times .7$
5. $.5 \times 7$	$.5 \times .07$	$.05 \times .07$
6. $.5 \times .7$	$.05 \times 7$	$5 \times .007$

How many digits should be recorded to the right of the decimal point in the simplest decimal numeral for each product below?

<i>a</i>	<i>b</i>	<i>c</i>
7. $.7 \times .9$	$27.1 \times .4$	$.07 \times 74.2$
8. $.06 \times .4$	6.1×1.7	$3.21 \times .08$
9. $1.4 \times .2$	16.9×9	$.75 \times .72$
10. $.11 \times .8$	$.08 \times .04$	$.7 \times .009$

Written Copy. Find the simplest decimal numeral for each product.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\begin{array}{r} .52 \\ \times .8 \\ \hline \end{array}$	$\begin{array}{r} .017 \\ \times 2.4 \\ \hline \end{array}$	$\begin{array}{r} .018 \\ \times .14 \\ \hline \end{array}$
2.	$\begin{array}{r} 2.5 \\ \times .9 \\ \hline \end{array}$	$\begin{array}{r} 942 \\ \times .04 \\ \hline \end{array}$	$\begin{array}{r} .176 \\ \times .3 \\ \hline \end{array}$
3.	$\begin{array}{r} .17 \\ \times 6 \\ \hline \end{array}$	$\begin{array}{r} 1.95 \\ \times .02 \\ \hline \end{array}$	$\begin{array}{r} 6.004 \\ \times .09 \\ \hline \end{array}$
4.	$\begin{array}{r} .24 \\ \times .56 \\ \hline \end{array}$	$\begin{array}{r} 3.02 \\ \times .04 \\ \hline \end{array}$	$\begin{array}{r} 17.23 \\ \times .13 \\ \hline \end{array}$
5.	$\begin{array}{r} 5.2 \\ \times 7 \\ \hline \end{array}$	$\begin{array}{r} 317 \\ \times 52 \\ \hline \end{array}$	$\begin{array}{r} 5.26 \\ \times 70 \\ \hline \end{array}$
6.	$\begin{array}{r} .37 \\ \times .4 \\ \hline \end{array}$	$\begin{array}{r} .176 \\ \times .9 \\ \hline \end{array}$	$\begin{array}{r} .512 \\ \times .26 \\ \hline \end{array}$

Can you do this? Since $3 \times 4 \times 2 = (3 \times 4) \times 2$ or 24, you can compute $.3 \times .4 \times .02$ as follows.

$$(.3 \times .4) \times .02 = .12 \times .02 = .0024$$

Does the pattern for determining the number of digits after the decimal point in the numeral for the product of two numbers also hold for the product of three numbers? Use that pattern to find the simplest decimal for each product below.

	<i>a</i>	<i>b</i>
1.	$.2 \times .2 \times .2$	$.7 \times .04 \times .12$
2.	$.03 \times .03 \times .03$	$.6 \times .121 \times .02$

Special Products

You already know the pattern for the first two products in the table below. Discover how it is extended for multiplication with decimals.

$$250 \times 10 = 2500$$

$$25 \times 10 = 250$$

$$2.5 \times 10 = 25$$

$$\text{Because } \frac{25}{\cancel{10}^1} \times \cancel{10}^1 = 25$$

$$.25 \times 10 = 2.5$$

$$\text{Because } \frac{25}{\cancel{100}^{10}} \times \cancel{10}^{10} = \frac{25}{10} = 2.5$$

$$.025 \times 10 = .25$$

$$\text{Because } \frac{25}{\cancel{1000}^{100}} \times \cancel{10}^{100} = \frac{25}{100} = .25$$

Compare the numerals 2.5, .25, and .025 with the product numerals 25, 2.5, and .25. What do you discover about placing the decimal point when multiplying by 10?

$$250 \times 100 = 25000$$

$$25 \times 100 = 2500$$

$$2.5 \times 100 = 250$$

$$.25 \times 100 = 25$$

$$.025 \times 100 = 2.5$$

$$250 \times 1000 = 250000$$

$$25 \times 1000 = 25000$$

$$2.5 \times 1000 = 2500$$

$$.25 \times 1000 = 250$$

$$.025 \times 1000 = 25$$

What pattern can you discover above for placing the decimal point when multiplying by 100? By 1000?

Oral State the simplest decimal for each product below.

Written Solve each equation below.

- | a | b |
|----------------------|-------------------|
| 1. 3.7×10 | $.37 \times 10$ |
| 2. $.42 \times 100$ | $.042 \times 100$ |
| 3. 5.7×1000 | $.57 \times 1000$ |

- | a | b |
|--------------------------|-------------------------|
| 1. $5.12 \times 10 = a$ | $c = 100 \times 17.51$ |
| 2. $10 \times .318 = b$ | $d = .0182 \times 100$ |
| 3. $4.7 \times 100 = e$ | $f = 1000 \times .27$ |
| 4. $.019 \times 100 = g$ | $h = 315.7 \times 1000$ |

Final Zeros in a Product Numeral

The pattern for placing the decimal point in a product numeral is used in the following example.

$$.25 \times .4 = .100$$

However, you have already learned that $.1 = .10 = .100$ and so on. Therefore, the sentence above can be changed as follows.

$$.25 \times .4 = .1$$

Deleting the final 0's to the right of the decimal point is equivalent to changing fractions to simplest form. This is shown below by changing $.25 \times .4$ to fractional form.

$$.25 \times .4 = \frac{25}{100} \times \frac{4}{10} = \frac{1}{4} \times \frac{2}{5} = \frac{1}{10} \text{ or } .1$$

Notice how the above ideas are used in the following multiplication examples.

$$\begin{array}{r} 5.24 \rightarrow 2 \text{ digits} \\ \times .25 \rightarrow 2 \text{ digits} \\ \hline 2620 \\ 10480 \\ \hline 1.3100 \leftarrow 4 \text{ digits} \end{array}$$

or
1.31

$$\begin{array}{r} 2000 \rightarrow 0 \text{ digits} \\ \times .03 \rightarrow 2 \text{ digits} \\ \hline 60.00 \leftarrow 2 \text{ digits} \end{array}$$

or
60

Oral Answer the following questions about $.25 \times 5.24$ above.

1. How can you tell where to place the decimal point in 1.3100?

2. Are 1.3100 and 1.31 both correct numerals for the product?

3. How could you show that 1.3100 is equal to 1.31?

Written Copy. Express each product as a decimal having the least possible number of digits.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$1.95 \times .02$	$.018 \times 1.5$	$400 \times .07$
2.	$.345 \times 16$	6.28×2.5	$74.4 \times .75$

Division with Decimals

The procedures for dividing whole numbers can be extended for division with decimals.

H	T	O	Ts	Hs	Ths
4	2	5	2		

$$4 \overline{) 252}$$

Round off the dividend to the nearest hundred. Think: $3H \div 4 < 1H$. Do you need to record a digit in hundreds place of the quotient numeral?

Now round off the dividend to the nearest ten. Think: $25T \div 4 \geq ___T$. What is the greatest number that $___$ can stand for? What is the first digit of the quotient numeral and where should it be recorded? What is the next digit of the quotient numeral and where should it be recorded?

H	T	O	Ts	Hs	Ths
4	2	5	2		

Round off the dividend to the nearest ten. Think: $3T \div 4 < 1T$. Do you need to record a digit in tens place of the quotient numeral?

Now round off the dividend to the nearest one. Think: $25 \div 4 \geq ___$. What is the greatest number that $___$ can stand for? What is the first digit of the quotient numeral and where should it be recorded? What is the next digit of the quotient numeral and where should it be recorded?

H	T	O	Ts	Hs	Ths
4	2	5	2		

Round off the dividend to the nearest one. Think: $3 \div 4 < 1$. Do you need to record a digit in ones place of the quotient numeral?

Now round off the dividend to the nearest tenth. Think: $25Ts \div 4 \geq ___Ts$. What is the greatest number that $___$ can stand for? What is the first digit of the quotient numeral and where should it be recorded? What is the next digit of the quotient numeral and where should it be recorded?

H	T	O	Ts	Hs	Ths
4	2	5	2		

Round off the dividend to the nearest tenth. Think: $3Ts \div 4 < 1Ts$. Why must you record a 0 in tenths place of the quotient numeral?

Now round off the dividend to the nearest hundredth. Think: $25Hs \div 4 \geq ___Hs$. What is the greatest number that $___$ can stand for? What is the first digit of the quotient numeral and where should it be recorded? What is the next digit of the quotient numeral and where should it be recorded?

Oral Explain how the quotient is found in the example below.

$$\begin{array}{r}
 .0 \\
 3 \overline{) .0429} \quad 0\text{T}s \div 3 \geq 0\text{T}s \\
 \underline{3} \\
 12 \\
 \underline{12} \\
 0
 \end{array}$$

$$\begin{array}{r}
 .01 \\
 3 \overline{) .0429} \quad 4\text{H}s \div 3 \geq 1\text{H}s \\
 \underline{3} \\
 12 \\
 \underline{12} \\
 0
 \end{array}$$

Record 1 in Hs place.

$$\begin{array}{r}
 .014 \\
 3 \overline{) .0429} \\
 \underline{3} \\
 12 \\
 \underline{12} \\
 0
 \end{array}$$

12Ths $\div 3 \geq 4\text{T}s$
Record 4 in Ths place.

$$\begin{array}{r}
 .0143 \\
 3 \overline{) .0429} \\
 \underline{3} \\
 12 \\
 \underline{12} \\
 09 \\
 \underline{9} \\
 0
 \end{array}$$

9TThs $\div 3 \geq 3$
Record 3 in TThs place.

Copy. Find each quotient.

<i>a</i>	<i>b</i>
7. $5 \overline{) 7.5}$	$7 \overline{) .112}$
8. $3 \overline{) .72}$	$5 \overline{) .085}$
9. $4 \overline{) 6.8}$	$3 \overline{) .0141}$
10. $6 \overline{) .078}$	$8 \overline{) .0072}$
11. $8 \overline{) 2.96}$	$6 \overline{) .0384}$
12. $7 \overline{) .266}$	$4 \overline{) 98.4}$
13. $2 \overline{) .0144}$	$9 \overline{) 64.8}$
14. $4 \overline{) 69.28}$	$8 \overline{) 90.08}$
15. $9 \overline{) 2.916}$	$6 \overline{) .00738}$
16. $6 \overline{) .0084}$	$7 \overline{) 14.945}$

Written Knowing that $224 \div 7 = 32$, and $522 \div 3 = 174$, write the decimal numeral for each quotient below.

<i>a</i>	<i>b</i>
1. $7 \overline{) 22.4}$	$7 \overline{) 2.24}$
2. $7 \overline{) .224}$	$7 \overline{) .0224}$
3. $7 \overline{) .00224}$	$7 \overline{) .000224}$
4. $3 \overline{) 52.2}$	$3 \overline{) .0522}$
5. $3 \overline{) 5.22}$	$3 \overline{) .00522}$
6. $3 \overline{) .522}$	$3 \overline{) .000522}$

Solve each problem below.

17. What was the average rainfall per hour if 1.74 inches of rain fell in 3 hours?

18. In 5 hours Don earned \$8.75. What was his hourly rate of pay?

19. In one week Tom's watch gained 19.6 seconds. How much did his watch gain in one day?

20. A pamphlet containing 8 sheets of paper was .032 inches thick. What was the thickness of each sheet of paper?

Division with Decimals

In division with decimals, a two-digit divisor is rounded off to the nearest ten when estimating the quotient. Due to such rounding off, dividing by ten as shown below is essential in finding a quotient.

$1 \div 10 = \frac{1}{10}$	$1 \div 1T = 1Ts$
$\frac{1}{10} \div 10 = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$	$1Ts \div 1T = 1Hs$
$\frac{1}{100} \div 10 = \frac{1}{100} \times \frac{1}{10} = \frac{1}{1000}$	$1Hs \div 1T = 1Ths$
$\frac{1}{1000} \div 10 = \frac{1}{1000} \times \frac{1}{10} = \frac{1}{10000}$	$1Ths \div 1T = 1TThs$

The procedures for dividing 1505 by 43 can be extended to dividing .1505 by 43 as explained below.

$\begin{array}{r} .00 \\ 43 \overline{) .1505} \end{array}$	Round off 43 to 4T. Think: $2Ts \div 4T \geq 0Hs$. This requires a 0 in Hs place, which also requires a 0 in Ts place.
$\begin{array}{r} .004 \\ 43 \overline{) .1505} \\ \underline{172} \end{array}$ $\begin{array}{r} .003 \\ 43 \overline{) .1505} \\ \underline{129} \\ 21 \end{array}$	Think: $16Hs \div 4T \geq 4Ths$. But $4Ths \times 43 = 172Ths$ and $172Ths > 150Ths$. Hence 4 is too great in Ths place, so use 3 in Ths place.
$\begin{array}{r} .0035 \\ 43 \overline{) .1505} \\ \underline{129} \\ 215 \\ \underline{215} \\ 0 \end{array}$	Think: $22Ths \div 4T \geq 5TThs$.

Think of finding the decimal numeral for the quotient $52 \overline{) .312}$. What will the Ts digit be? What will the Hs digit be? What will the Ths digit be? How did you determine the Ths digit?

Oral Study each example below and answer the questions that follow it.

$$72 \overline{)26.64} \quad 27 \div 7T \geq __Ts$$

1. How is the sentence $27 \div 7T \geq __Ts$ obtained?

2. What numeral should replace the $__$?

3. Where will that numeral be recorded in the quotient numeral?

$$\begin{array}{r} .0 \\ 72 \overline{)2.664} \end{array} \quad \begin{array}{l} 3 \div 7T \geq 0Ts \\ 27Ts \div 7T \geq __Hs \end{array}$$

4. How is $27Ts \div 7T \geq __Hs$ obtained?

5. What numeral should replace the $__$?

6. Where will that numeral be recorded in the quotient numeral?

$$\begin{array}{r} .00 \\ 72 \overline{)2.664} \end{array} \quad \begin{array}{l} 3Ts \div 7T \geq 0Hs \\ 27Hs \div 7T \geq __Ths \end{array}$$

7. How is $27Hs \div 7T \geq __Ths$ obtained?

8. What numeral should replace the $__$?

9. Where will that numeral be recorded in the quotient numeral?

Knowing that $234 \div 13 = 18$, state the decimal numeral for each quotient below.

a	b	c
10. $13 \overline{)23.4}$	$13 \overline{)2.34}$	$13 \overline{).234}$

Written Copy. Find each quotient.

a

b

1. $27 \overline{).216}$

$32 \overline{).0544}$

2. $46 \overline{)1.058}$

$13 \overline{)1.508}$

3. $25 \overline{)1.25}$

$17 \overline{).782}$

4. $14 \overline{)25.2}$

$34 \overline{)435.2}$

5. $73 \overline{).3796}$

$26 \overline{)63.18}$

6. $59 \overline{).413}$

$42 \overline{).0756}$

7. $86 \overline{).0602}$

$64 \overline{)940.8}$

8. $54 \overline{)1.458}$

$51 \overline{)2759.1}$

9. $12 \overline{)45.6}$

$41 \overline{)378.02}$

10. $23 \overline{).1472}$

$89 \overline{)6372.4}$

Solve each problem below.

11. A power boat traveled 3909.75 feet in 25 seconds. How far did it travel in one second?

12. A pamphlet containing 32 sheets of paper was .128 inches thick. What was the thickness of each sheet of paper?

13. John worked 38 hours last week. His total earnings were \$66.50. What was his hourly rate of pay?

14. What is the average of 3.263, 17.296, and 22.41?

Division with Decimals

You have learned how to find a quotient when

- the divisor is a whole number, and
- the dividend is a rational number named by a decimal.

Many problems require a divisor that is not a whole number. There is a way of changing a division expression in which the divisor is not a whole number to an equivalent expression in which the divisor is a whole number.

$$.035 \div .07 = \frac{35}{1000} \div \frac{7}{100} = \frac{\overset{5}{\cancel{35}}}{\underset{10}{\cancel{1000}}} \times \frac{\overset{1}{\cancel{100}}}{\underset{1}{\cancel{7}}} = \frac{5}{10} = .5$$

$$.35 \div .7 = \frac{35}{100} \div \frac{7}{10} = \frac{\overset{5}{\cancel{35}}}{\underset{10}{\cancel{100}}} \times \frac{\overset{1}{\cancel{10}}}{\underset{1}{\cancel{7}}} = \frac{5}{10} = .5$$

$$3.5 \div 7 = \frac{35}{10} \div \frac{7}{1} = \frac{\overset{5}{\cancel{35}}}{\underset{10}{\cancel{10}}} \times \frac{\overset{1}{\cancel{1}}}{\underset{1}{\cancel{7}}} = \frac{5}{10} = .5$$

All of the above division numerals name the same number. The quotient is the same for each expression below.

$$.07 \overline{) .035}$$

$$.7 \overline{) .35}$$

$$7 \overline{) 3.5}$$

Which of the divisions above have you already studied? Therefore, it seems wise to change any of the others so that the divisor is a whole number. You merely have to use the identity number of multiplication to do that.

$$.2 \overline{) 1.4} \quad \frac{1.4}{.2} = \frac{1.4}{.2} \times 1 = \frac{1.4}{.2} \times \frac{10}{10} = \frac{14}{2} \quad 2 \overline{) 14}$$

Why use $\frac{10}{10}$?

Therefore, $.2 \overline{) 1.4}$ has the same quotient as $2 \overline{) 14}$.

$$.02 \overline{) 1.4} \quad \frac{1.4}{.02} = \frac{1.4}{.02} \times 1 = \frac{1.4}{.02} \times \frac{100}{100} = \frac{140}{2} \quad 2 \overline{) 140}$$

Why use $\frac{100}{100}$?

Therefore, $.02 \overline{) 1.4}$ has the same quotient as $2 \overline{) 140}$.

$$.002 \overline{)1.4} \longrightarrow \frac{1.4}{.002} = \frac{1.4}{.002} \times 1 = \frac{1.4}{.002} \times \frac{1000}{1000} = \frac{1400}{2} \longrightarrow 2 \overline{)1400}$$

Why use $\frac{1000}{1000}$?

Therefore, $.002 \overline{)1.4}$ has the same quotient as $2 \overline{)1400}$.

Is the effect of multiplying a fractional number by 1 the same as multiplying both the dividend and the divisor by the same power of ten? How can you decide which power of ten to use?

Oral Change each division expression below to another division expression so that the quotient remains the same but the divisor is a whole number.

<i>a</i>	<i>b</i>	<i>c</i>
1. $.2 \overline{).8}$	$.8 \overline{)4.8}$	$.36 \overline{)2.88}$
2. $.2 \overline{).08}$	$.12 \overline{).048}$	$.52 \overline{)312}$
3. $.02 \overline{).08}$	$1.5 \overline{).75}$	$.123 \overline{)73.8}$

Explain how the quotient is found in each example below.

4. $1.2 \overline{)4.56}$	$\begin{array}{r} 3.8 \\ 12 \overline{)45.6} \\ \underline{36} \\ 96 \\ \underline{96} \\ 0 \end{array}$
5. $.34 \overline{)3.128}$	$\begin{array}{r} 9.2 \\ 34 \overline{)312.8} \\ \underline{306} \\ 68 \\ \underline{68} \\ 0 \end{array}$

Written Change each of the following so that the divisor is a whole number. Then find each quotient.

<i>a</i>	<i>b</i>
1. $.9 \overline{).072}$	$.09 \overline{).72}$
2. $.08 \overline{).056}$	$.008 \overline{).0056}$
3. $1.2 \overline{)1.56}$	$.012 \overline{)1.56}$
4. $3.6 \overline{)24.84}$	$.36 \overline{)248.4}$
5. $.014 \overline{).476}$	$.14 \overline{).0476}$
6. $8.3 \overline{)2.0169}$	$.83 \overline{)20.169}$

Another way A pattern for placing the decimal point in a quotient numeral is evident in these examples.

$$\begin{array}{l} .08 \overline{)2.4} \longrightarrow 8 \overline{)240} \\ 2.134 \overline{)1.4938} \longrightarrow 2134 \overline{)1493.8} \\ .002 \overline{)40.} \longrightarrow 2 \overline{)40000} \end{array}$$

Use this pattern to find the quotient numerals for *Oral* 1–3.

Changing Fractions to Decimals

You already know that $\frac{1}{2}$ can be changed to a decimal as shown below.

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{5}{5} = \frac{5}{10} = .5$$

You also know that the fraction line can be interpreted as a division symbol.

$$\frac{1}{2} \text{ can be interpreted as } 2 \overline{)1}.$$

Therefore, it must be possible to develop a division algorithm for $\frac{1}{2}$ in the form $2 \overline{)1}$ such that the quotient is .5. Recall that 1 can be expressed as a decimal in any of the following ways.

$$1 = 1.0 = 1.00 = 1.000 = 1.0000, \text{ and so on.}$$

Therefore, the quotient should be the same for each of the following divisions. What is that quotient?

$$2 \overline{)1}$$

$$2 \overline{)1.0}$$

$$2 \overline{)1.00}$$

$$2 \overline{)1.000}$$

Study how the above ideas are used in expressing $\frac{3}{16}$ as a decimal. Since $16 > 3$, rename 3 as 3.0 and carry out the division procedure as usual.

$$\begin{array}{r} 16 \overline{)3.0} \\ \underline{16} \\ 14 \end{array}$$

Since the difference is not 0, name 3 by 3.00.

$$\begin{array}{r} 16 \overline{)3.00} \\ \underline{16} \\ 140 \\ \underline{128} \\ 12 \end{array}$$

Since the difference is not 0, name 3 by 3.000.

$$\begin{array}{r} 16 \overline{)3.000} \\ \underline{16} \\ 140 \\ \underline{128} \\ 120 \\ \underline{112} \\ 8 \end{array}$$

Since the difference is not 0, name 3 by 3.0000.

$$\begin{array}{r} 16 \overline{)3.0000} \\ \underline{16} \\ 140 \\ \underline{128} \\ 120 \\ \underline{112} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

The difference is 0, so the decimal terminates.

Oral Explain how $\frac{3}{4}$ and $\frac{1}{8}$ are changed to decimals below.

$$\begin{array}{r} .75 \\ 4 \overline{) 3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

$$\begin{array}{r} .125 \\ 8 \overline{) 1.000} \\ \underline{8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Answer the following questions.

1. How can you tell from $\frac{3}{4}$ and $\frac{1}{8}$ that their equivalent decimals are terminating decimals?

2. Is it necessary to determine how many 0's to place after the decimal point in the dividend numeral before dividing?

3. How does the difference at each step of the procedure show whether or not another 0 should be placed to the right of the decimal point in the dividend numeral?

Written Use division to change each fraction below to an equivalent decimal.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{4}{5}$	$\frac{3}{20}$	$\frac{7}{25}$
2. $\frac{21}{25}$	$\frac{3}{8}$	$\frac{1}{16}$
3. $\frac{7}{20}$	$\frac{1}{8}$	$\frac{5}{16}$
4. $\frac{6}{12}$	$\frac{9}{15}$	$\frac{8}{32}$

5. $\frac{11}{50}$

$\frac{7}{8}$

$\frac{33}{75}$

6. $\frac{3}{4}$

$\frac{21}{28}$

$\frac{14}{35}$

Change each fraction below to an equivalent decimal by using the most convenient method.

<i>a</i>	<i>b</i>	<i>c</i>
7. $\frac{9}{30}$	$\frac{17}{50}$	$\frac{27}{72}$
8. $\frac{26}{65}$	$\frac{17}{250}$	$\frac{39}{40}$
9. $\frac{9}{40}$	$\frac{16}{80}$	$\frac{82}{250}$
10. $\frac{13}{25}$	$\frac{19}{80}$	$\frac{13}{16}$
11. $\frac{48}{500}$	$\frac{19}{76}$	$\frac{42}{56}$

Solve each equation below.

<i>a</i>	<i>b</i>
12. $n = 39 \div 52$	$12 \div 80 = n$
13. $8n = 5$	$27 = 45n$

Can you do this? You already know that $\frac{3}{8} = \frac{3 \times 1}{8} = 3 \times \frac{1}{8}$.

Since $\frac{1}{8} = .125$, you also know that $\frac{3}{8} = 3 \times .125 = .375$.

Also, $\frac{4}{8} = 4 \times .125 = .500$ or $.5$, and $\frac{5}{8} = 5 \times .125 = .625$.

Use this idea to find the decimal equivalent of each fraction below.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
1. $\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{7}{5}$
2. $\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{9}{16}$	$\frac{13}{16}$

Repeating Decimals

Recall that a fraction in simplest form whose denominator contains a prime factor other than 2 or 5 cannot be changed to a terminating decimal. By using division such fractions can be expressed as repeating decimals.

Let us interpret $\frac{1}{3}$ as $1 \div 3$ or $3 \overline{)1}$ and carry out the division procedure.

$$\begin{array}{r} .33 \\ 3 \overline{)1.00} \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

If you place another 0 in the dividend numeral, would you be able to find another digit in the quotient? Without dividing, what would that digit be? Would the decimal for the quotient ever terminate or end?

Hence, $\frac{1}{3} = .333\ldots$ where $.333\ldots$ denotes a non-ending sequence of 3's. A simpler symbol for this idea is to draw a bar over the digit or digits that are repeated.

$\overline{.3}$ means $.33333\ldots$ and so on indefinitely.

A decimal like $\overline{.3}$ is called a **repeating decimal**. The digit or digits that are repeated is called the **period**.

The period in $\overline{.3}$ contains but one digit.

Since $\frac{1}{3}$ names a rational number, so does $\overline{.3}$. To avoid confusing $\overline{.3}$ with other rational numbers, study the following and observe the difference in each case.

$$.3 = \frac{3}{10} \text{ and } \overline{.3} = \frac{1}{3}.$$

$$\text{Hence, } \frac{1}{3} - \frac{3}{10} = \frac{10}{30} - \frac{9}{30} = \frac{1}{30}.$$

$$.33 = \frac{33}{100} \text{ and } \overline{.3} = \frac{1}{3}.$$

$$\text{Hence, } \frac{1}{3} - \frac{33}{100} = \frac{100}{300} - \frac{99}{300} = \frac{1}{300}.$$

$$.333 = \frac{333}{1000} \text{ and } \overline{.3} = \frac{1}{3}.$$

$$\text{Hence, } \frac{1}{3} - \frac{333}{1000} = \frac{1000}{3000} - \frac{999}{3000} = \frac{1}{3000}.$$

Which of the terminating decimals, .3, .33 or .333, names a number nearest to $\frac{1}{3}$ or $\frac{1}{3}$?

Observe how $\frac{5}{6}$ and $\frac{3}{11}$ are changed to repeating decimals.

$$\begin{array}{r} .833 \\ 6 \overline{) 5.00} \\ \underline{48} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

$$\begin{array}{r} .2727 \\ 11 \overline{) 3.0000} \\ \underline{22} \\ 80 \\ \underline{77} \\ 30 \\ \underline{22} \\ 80 \\ \underline{77} \\ 3 \end{array}$$

Hence, $\frac{5}{6} = .8\overline{3}$. What does the bar over the 3 in $.8\overline{3}$ mean? Also, $\frac{3}{11} = .\overline{27}$. What does $.\overline{27}$ mean?

The following statement can be proved, but we have only given examples for it.

Every rational number can be named by either a terminating decimal or a repeating decimal.

Oral Answer the questions below.

1. How many digits are there in the period in $.8\overline{3}$?
2. How does the repetition of the difference 2 indicate that the quotient numeral for $\frac{5}{6}$ will be a repeating decimal?
3. How many digits are there in the period in $.\overline{27}$?
4. In the division process for $\frac{3}{11}$, what is the signal that tells you that the resulting decimal will be a repeating decimal?

Written Change each fraction below to a repeating decimal.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{1}{15}$
2.	$\frac{5}{11}$	$\frac{4}{33}$	$\frac{1}{18}$	$\frac{7}{9}$
3.	$\frac{1}{12}$	$\frac{5}{9}$	$\frac{9}{11}$	$\frac{1}{7}$

Can you do this? Arrange the numerals in each list below so they name numbers from least to greatest.

1. $.\overline{24}$, $.24$, $.\overline{2\overline{4}}$
2. $.\overline{234}$, $.\overline{23\overline{4}}$, $.234$, $.\overline{23\overline{4}}$

Approximation

Sometimes a terminating decimal contains more digits than are practical or convenient to use. The same is true for all repeating decimals. To make either kind of decimal useful for practical purposes, an approximation procedure similar to that used for whole numbers is employed.

$$\begin{array}{r}
 .294117 \\
 17 \overline{) 5.000000} \\
 \underline{34} \\
 160 \\
 \underline{153} \\
 70 \\
 \underline{68} \\
 20 \\
 \underline{17} \\
 30 \\
 \underline{17} \\
 130 \\
 \underline{119} \\
 11
 \end{array}$$

$$\begin{array}{r}
 .140625 \\
 64 \overline{) 9.000000} \\
 \underline{64} \\
 260 \\
 \underline{256} \\
 400 \\
 \underline{384} \\
 160 \\
 \underline{128} \\
 320 \\
 \underline{320} \\
 0
 \end{array}$$

By rounding off either quotient above, you name another rational number which is an *approximation* to the rational number named by the repeating decimal or by the lengthy terminating decimal. Observe below that the more digits used to name an approximation for $\frac{5}{17}$, the closer that approximation is to the rational number being approximated.

Approximation

Difference

To the nearest tenth: .3

$$\frac{3}{10} - \frac{5}{17} = \frac{51}{170} - \frac{50}{170} = \frac{1}{170}$$

To the nearest hundredth: .29

$$\frac{5}{17} - \frac{29}{100} = \frac{500}{1700} - \frac{493}{1700} = \frac{7}{1700}$$

To the nearest thousandth: .294

$$\frac{5}{17} - \frac{294}{1000} = \frac{5000}{17000} - \frac{4998}{17000} = \frac{2}{17000}$$

How would you approximate .140625 to the nearest tenth?
To the nearest hundredth? To the nearest thousandth?

Oral Answer questions 1–5 about the example below.

$$\begin{array}{r} .1388 \\ 36 \overline{) 5.0000} \\ \underline{36} \\ 140 \\ \underline{108} \\ 320 \\ \underline{288} \\ 320 \\ \underline{288} \\ 32 \end{array}$$

1. What would be the next digit in the quotient numeral? How did you decide?

2. The process above shows how to find the repeating decimal for what fraction?

3. How would you state the quotient by using the bar notation to show the period?

4. To the nearest thousandth, what is the approximation for the quotient?

5. To the nearest hundredth, what is the approximation for the quotient?

State an approximation to the nearest hundredth for each number below.

	<i>a</i>	<i>b</i>	<i>c</i>
6.	3.2874	$7.\overline{4}$	$53.\overline{09}$
7.	31.0482	$17.3\overline{7}$	$8.1\overline{26}$

Written Find a decimal approximation to the nearest tenth for each rational number below.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$\frac{11}{12}$	$\frac{9}{17}$	$\frac{24}{31}$	$\frac{19}{47}$
2.	$\frac{17}{26}$	$\frac{24}{29}$	$\frac{39}{74}$	$\frac{15}{83}$
3.	$\frac{85}{11}$	$\frac{91}{49}$	$\frac{111}{77}$	$\frac{241}{69}$
4.	$\frac{39}{15}$	$\frac{51}{71}$	$\frac{138}{129}$	$\frac{621}{30}$

Find a decimal approximation to the nearest hundredth for each rational number below.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
5.	$\frac{7}{12}$	$\frac{11}{17}$	$\frac{29}{31}$	$\frac{13}{47}$
6.	$\frac{19}{26}$	$\frac{21}{29}$	$\frac{29}{74}$	$\frac{18}{89}$
7.	$\frac{71}{26}$	$\frac{29}{24}$	$\frac{47}{39}$	$\frac{83}{15}$
8.	$\frac{58}{11}$	$\frac{19}{94}$	$\frac{77}{13}$	$\frac{147}{62}$

Tell why Why does $\frac{9}{12}$ have a terminating decimal equivalent, while $\frac{8}{12}$ and $\frac{10}{12}$ do not?

Can you do this? Find a repeating decimal for $\frac{1}{11}$, for $\frac{2}{11}$, and for $\frac{3}{11}$. Study the digits in the period of each repeating decimal to discover a pattern. Predict the repeating decimal for $\frac{4}{11}$, for $\frac{5}{11}$, for $\frac{6}{11}$, for $\frac{7}{11}$, and for $\frac{8}{11}$. Check your predictions by division.

Fractions and Equivalent Decimals

Whenever a repeating decimal is rounded off to provide a useful approximation, some error is introduced. To avoid such errors, we can interrupt the division process at any stage and state the quotient by using a mixed numeral.

Suppose you change $\frac{3}{7}$ to an equivalent decimal.

$$\begin{array}{r} .4 \\ 7 \overline{) 3.0} \\ \underline{28} \\ 2 \end{array}$$

Stop. Express the result as $.4\frac{2}{7}$.

$$\begin{array}{r} .42 \\ 7 \overline{) 3.00} \\ \underline{28} \\ 20 \\ \underline{14} \\ 6 \end{array}$$

Stop. Express the result as $.42\frac{6}{7}$.

$$\begin{array}{r} .428 \\ 7 \overline{) 3.000} \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

Stop. Express the result as $.428\frac{4}{7}$.

Just as we call a numeral like $7\frac{1}{2}$ a mixed numeral, let us call a numeral like $.42\frac{6}{7}$ a *mixed decimal numeral* or simply a *mixed decimal*.

By changing the above results back to $\frac{3}{7}$, you can discover the meaning of a fraction in a mixed decimal.

$$.4\frac{2}{7} = .4 + \left(\frac{2}{7} \times .1\right) = \frac{4}{10} + \left(\frac{2}{7} \times \frac{1}{10}\right) = \frac{28}{70} + \frac{2}{70} = \frac{30}{70} = \frac{3}{7}$$

$$.42\frac{6}{7} = .42 + \left(\frac{6}{7} \times .01\right) = \frac{42}{100} + \left(\frac{6}{7} \times \frac{1}{100}\right) = \frac{294}{700} + \frac{6}{700} = \frac{300}{700} = \frac{3}{7}$$

$$.428\frac{4}{7} = .428 + \left(\frac{4}{7} \times .001\right) = \frac{428}{1000} + \left(\frac{4}{7} \times \frac{1}{1000}\right) = \frac{2996}{7000} + \frac{4}{7000} = \frac{3000}{7000} = \frac{3}{7}$$

A fraction in a mixed decimal always refers to the place value of the preceding digit.

In $.6\frac{2}{3}$, $\frac{2}{3}$ stands for $\frac{2}{3} \times \frac{1}{10}$ or $\frac{2}{30}$ or $\frac{1}{15}$.

In $.06\frac{2}{3}$, what does $\frac{2}{3}$ stand for?

In $.006\frac{2}{3}$, what does $\frac{2}{3}$ stand for?

The same technique can be used on terminating decimals as shown below for $\frac{1}{8}$.

$$\begin{array}{r} .1 \\ 8 \overline{) 1.0} \\ \underline{8} \\ 2 \end{array}$$

$$\frac{1}{8} = .1\frac{2}{8} = .1\frac{1}{4}$$

$$\begin{array}{r} .12 \\ 8 \overline{) 1.00} \\ \underline{8} \\ 20 \\ \underline{16} \\ 4 \end{array}$$

$$\frac{1}{8} = .12\frac{4}{8} = .12\frac{1}{2}$$

$$\begin{array}{r} .125 \\ 8 \overline{) 1.000} \\ \underline{8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

$$\text{Therefore, } .125 = .1\frac{1}{4} = .12\frac{1}{2}.$$

Oral State the number named by the fraction in each of the following mixed decimals.

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|-------------------|-------------------|-------------------|
| 1. | $1.5\frac{3}{4}$ | $.07\frac{1}{3}$ | $.40\frac{3}{7}$ |
| 2. | $.016\frac{1}{2}$ | $17.3\frac{1}{4}$ | $9.18\frac{1}{5}$ |

Written Change each mixed decimal below to a terminating decimal.

- | | <i>a</i> | <i>b</i> | <i>c</i> | <i>d</i> |
|----|-----------------|------------------|-------------------|-------------------|
| 1. | $.5\frac{1}{4}$ | $.17\frac{1}{2}$ | $.32\frac{2}{5}$ | $.346\frac{1}{4}$ |
| 2. | $.3\frac{3}{4}$ | $.7\frac{3}{5}$ | $.214\frac{1}{2}$ | $.487\frac{3}{4}$ |
| 3. | $.7\frac{1}{8}$ | $.13\frac{1}{5}$ | $.68\frac{3}{8}$ | $.72\frac{3}{5}$ |

Write an open sentence for each problem. Solve the open sentence. Answer the problem.

4. Mr. Gray bought 17.5 gallons of gasoline. His bill was \$5.74. What was the price per gallon?

5. Sea water weighs approximately 8.58 pounds per gallon. What is the approximate weight of 325 gallons of sea water?

6. The length of a rectangle is 6 times greater than its width. If its width is 4.15 cm., what is its length?

7. One inch is equivalent to approximately 2.54 centimeters. Approximately how many inches are equivalent to 33.02 centimeters?

Tell how Show that the numerals in each row below name the same number.

- $.2\frac{1}{20}$, $.20\frac{1}{2}$, .205
- $.3\frac{3}{4}$, $.37\frac{1}{2}$, .375
- $1\frac{1}{8}$, $1.1\frac{1}{4}$, 1.125
- .1625, $.162\frac{1}{2}$, $.16\frac{1}{4}$, $.1\frac{5}{8}$
- .8375, $.837\frac{1}{2}$, $.83\frac{3}{4}$, $.8\frac{3}{8}$

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. If the denominator has prime factors of only 2 or 5, or both 2 and 5, the fraction can be changed to a terminating decimal. (160–162)

2. If a fraction is in simplest form and the denominator has a prime factor other than 2 or 5, the fraction cannot be changed to a terminating decimal, only to a repeating decimal. (162, 180)

3. Addition and subtraction with decimals are similar to adding and subtracting whole numbers. (164–167)

4. Multiplication and division with decimals are similar to multiplying and dividing whole numbers, except for placing the decimal point in the product or quotient numeral. (168–177)

5. Every rational number can be named by either a terminating or a repeating decimal. (181)

Words to Know

1. Decimal numerals (155)
2. Terminating decimals (162)
3. Repeating decimals, period (180)
4. Approximation (182)

Questions to Discuss

1. How would you read each of the following: 5.7, .37, 172.067? (155–157)

2. How would you change each of the following to an equivalent decimal: $\frac{2}{5}$, $\frac{21}{60}$, $\frac{9}{40}$? (160–161)

3. How can you tell when a fraction can be changed to a terminating decimal? (162)

4. Knowing that $17 \times 35 = 595$, how can you tell where to place the decimal point in the product numeral for $.17 \times 3.5$? (168)

5. How would you divide 4.08 by 12? (174)

6. How would you divide 8.657 by .27? (176)

7. What does $.1\overline{6}$ mean? (180)

Written Practice

Solve each equation below.

1. $17.04 + 2.987 = n$ (164)
2. $53.17 - 26.34 = r$ (166)
3. $15.2 \times .46 = t$ (168)
4. $37.7 \div 29 = s$ (172)
5. $14.58 \div 5.4 = v$ (176)

Self-Evaluation

Part 1 Change each decimal below to an equivalent fraction in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. .8	.35	.125
2. .9	.75	.005
3. .3	.64	.675

Change each fraction below to an equivalent decimal.

<i>a</i>	<i>b</i>	<i>c</i>
4. $\frac{4}{5}$	$\frac{7}{25}$	$\frac{5}{16}$
5. $\frac{3}{4}$	$\frac{16}{20}$	$\frac{7}{125}$
6. $\frac{7}{14}$	$\frac{11}{40}$	$\frac{9}{50}$

Part 2 Copy. Compute each sum or difference.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\begin{array}{r} 82.6 \\ +5.3 \\ \hline \end{array}$	$\begin{array}{r} 9.47 \\ +.056 \\ \hline \end{array}$	$\begin{array}{r} 19.106 \\ +7.926 \\ \hline \end{array}$
2. $\begin{array}{r} 1.026 \\ 5.41 \\ +.378 \\ \hline \end{array}$	$\begin{array}{r} 7.26 \\ .7 \\ +.69 \\ \hline \end{array}$	$\begin{array}{r} 32.74 \\ 8.506 \\ +7.371 \\ \hline \end{array}$
3. $\begin{array}{r} 14.9 \\ -6.2 \\ \hline \end{array}$	$\begin{array}{r} 8.81 \\ -3.78 \\ \hline \end{array}$	$\begin{array}{r} 14.79 \\ -8.463 \\ \hline \end{array}$
4. $\begin{array}{r} 16.04 \\ -3.88 \\ \hline \end{array}$	$\begin{array}{r} 18.67 \\ -9.8 \\ \hline \end{array}$	$\begin{array}{r} 3.019 \\ -.082 \\ \hline \end{array}$

Part 3 Copy. Compute each product or quotient.

<i>a</i>	<i>b</i>
1. $\begin{array}{r} 171.4 \\ \times 3.8 \\ \hline \end{array}$	$\begin{array}{r} 6.089 \\ \times .104 \\ \hline \end{array}$
2. $\begin{array}{r} 1.004 \\ \times .02 \\ \hline \end{array}$	$\begin{array}{r} 65.013 \\ \times 5.07 \\ \hline \end{array}$
3. $\begin{array}{r} 32.17 \\ \times 5.4 \\ \hline \end{array}$	$\begin{array}{r} 87.307 \\ \times .056 \\ \hline \end{array}$

4. $14 \overline{)2.66}$	$.157 \overline{).4239}$
5. $3.6 \overline{)24.48}$	$.076 \overline{).17936}$
6. $24 \overline{).0576}$	$.14 \overline{)33.852}$

Part 4 Use the bar notation, as in $.1\overline{4}$, to change each fraction below to a repeating decimal.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{2}{3}$	$\frac{5}{9}$	$\frac{7}{11}$
2. $\frac{8}{33}$	$\frac{1}{5}$	$\frac{1}{24}$

Use division to obtain an approximation to the nearest hundredth for each fraction below.

<i>a</i>	<i>b</i>	<i>c</i>
3. $\frac{5}{7}$	$\frac{2}{13}$	$\frac{7}{8}$
4. $\frac{9}{14}$	$\frac{1}{18}$	$\frac{21}{47}$

Midyear Review

Numeration

1. Why is the decimal numeration system called a positional system? (12)

2. How would you group the objects in a set to name its number by a base-five numeral? By a base-seven numeral? By a decimal numeral? (14–20)

3. How would you change 312_{five} to a decimal numeral? (16)

4. How would you change 64 to a base-five numeral? To a base-seven numeral? (22–24)

Mathematical Sentences

1. Which operation would you do first in $(72 \div 8) + 16$? In $3 \times 4 + 5$? In $[8 + (5 \times 2)] \div 3$? (30)

2. What is a variable? (32)

3. How can you tell whether a sentence is an open sentence? A closed sentence? An equality? An inequality? (34–37)

4. What does it mean to solve an open sentence? (38)

5. Why is $17 < 17$ a false sentence but $17 \leq 17$ a true sentence? (42)

6. What operation is understood in $13k$ and ab ? (44)

Whole Numbers

Explain what is meant by each sentence below. Then give an example for each sentence.

1. Addition has an identity number. (52)

2. Addition is commutative. (52)

3. Addition is associative. (53)

4. Addition and subtraction are inverse operations. (54)

5. Multiplication is commutative. (58)

6. Multiplication is associative. (59)

7. Multiplication has an identity number. (60)

8. Multiplication distributes over addition. (62)

9. Multiplication and division are inverse operations. (66)

Integers

Tell how you would solve each open sentence below.

1. $+3 + -7 = x$ (94)

2. $+5 - +8 = t$ (100)

3. $-8 \times +3 = a$ (102)

4. $+21 \div -7 = c$ (104)

Rational Numbers

1. How would you define a rational number? (110)
2. How can you find the product of $\frac{3}{5}$ and $\frac{2}{7}$? (112)
3. How can you find the greatest common factor of two numbers? (116)
4. When is a fraction in simplest form? (118)
5. When is a mixed numeral in simplest form? (123)
6. How can you express $\frac{3}{8} \div \frac{1}{4}$ as a single fraction? (126)
7. How can you determine whether two fractions are equivalent? (134)
8. How can you determine whether two fractions are not equivalent? (136)
9. How can you find the least common multiple of two numbers? (138)
10. What is meant by the least common denominator of two or more rational numbers? (140)
11. How can you find the sum of $\frac{5}{6}$ and $\frac{3}{4}$? (142)
12. How can you express $\frac{3}{4} - \frac{2}{5}$ as a single fraction? (146)
13. How would you rename 7 to solve $7 - 3\frac{4}{5} = n$? (148)

Decimals

1. How would you read the decimal 327.204? (157)
2. How would you change .13 to a fraction? 34.75 to a mixed numeral? (158)
3. How can you tell whether or not a fraction can be changed to a terminating decimal? (160-162)
4. How would you change $\frac{27}{36}$ to a terminating decimal? (161)
5. How is addition with decimals like addition of whole numbers? (164)
6. To solve $1.15 \times .7 = n$ you can find the product of 115 and 7. How do you decide where to place the decimal point in the product numeral? (168)
7. How can you change $.02\overline{)1.4}$ to another division expression that has the same quotient and a whole number as the divisor? (176)
8. What is meant by a repeating decimal? (180)
9. What is the period in $\overline{.21}$? (180)
10. What rational number is an approximation for .24057 to the nearest thousandth? (182)
11. Does the $\frac{1}{6}$ in $.42\frac{1}{6}$ refer to the thousandth place or to the hundredth place? (184)

Midyear Tests

Test 1 Find the simplest decimal numeral for each numeral below.

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|---------------------|----------------------|--------------------|
| 1. | XCVI | CCIX | MCDII |
| 2. | 23 _{five} | 15 _{seven} | 11 _{two} |
| 3. | 134 _{five} | 246 _{seven} | 101 _{two} |

Copy. Insert both () and [] in each expression so that it becomes a name for the number indicated after it.

- | | | |
|----|---------------------|------------|
| 4. | $24 - 8 + 4 \div 2$ | Number: 6 |
| 5. | $24 - 8 + 4 \div 2$ | Number: 10 |
| 6. | $24 - 8 + 4 \div 2$ | Number: 14 |
| 7. | $24 - 8 + 4 \div 2$ | Number: 18 |

Test 2 Copy. Solve each open sentence.

- | | <i>a</i> | <i>b</i> |
|----|------------------------------------|---------------------------------------|
| 1. | $23 + a = 51$ | $(18 - 5) + 5 = n$ |
| 2. | $x - 17 = 32$ | $32 \div (2 + 6) = k$ |
| 3. | $6b = 42$ | $(19 - c) \div 5 = 0$ |
| 4. | $\frac{n}{12} = 156$ | $(5 + n) \times 3 = 27$ |
| 5. | $+7 + -5 = n$ | $+81 \div -7 = r$ |
| 6. | $d = \frac{7}{8} \div \frac{2}{3}$ | $-15 \times +3 = s$ |
| 7. | $-3 - -8 = a$ | $t = \frac{2}{5} \times 3\frac{1}{3}$ |

$$8. \quad \frac{5}{7} = \frac{n}{81} \qquad \frac{n}{14} = \frac{45}{21}$$

$$9. \quad \frac{7}{12} + \frac{4}{5} = a \qquad m = 5\frac{1}{2} - 3\frac{3}{7}$$

$$10. \quad .13 + b = .76 \qquad (.7 - .3) \times .2 = d$$

Test 3 Copy. Compute each sum, difference, product, or quotient.

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|---|--|--|
| 1. | $\begin{array}{r} 326 \\ +472 \\ \hline \end{array}$ | $\begin{array}{r} 5467 \\ +935 \\ \hline \end{array}$ | $\begin{array}{r} 8045 \\ +7286 \\ \hline \end{array}$ |
| 2. | $\begin{array}{r} 873 \\ -522 \\ \hline \end{array}$ | $\begin{array}{r} 6024 \\ -376 \\ \hline \end{array}$ | $\begin{array}{r} 6703 \\ -3728 \\ \hline \end{array}$ |
| 3. | $\begin{array}{r} 423 \\ \times 6 \\ \hline \end{array}$ | $\begin{array}{r} 857 \\ \times 34 \\ \hline \end{array}$ | $\begin{array}{r} 5046 \\ \times 428 \\ \hline \end{array}$ |
| 4. | $\begin{array}{r} 3.2 \\ \times .7 \\ \hline \end{array}$ | $\begin{array}{r} 7.04 \\ \times .8 \\ \hline \end{array}$ | $\begin{array}{r} 65.21 \\ \times .47 \\ \hline \end{array}$ |

$$5. \quad 6 \overline{)78} \qquad 41 \overline{)1476} \qquad 524 \overline{)56068}$$

$$6. \quad 4 \overline{)5.2} \qquad .8 \overline{)1.84} \qquad .36 \overline{)3.924}$$

Solve each problem below.

7. Betty sold 7 tickets on Monday and 15 on Tuesday. She collected 25¢ for each ticket. How much money did she collect for all the tickets?

8. John had to make a 385-mile trip. After driving for 4 hours at an average speed of $57\frac{1}{2}$ miles per hour, how many miles did he still have to drive?

Chapter 8

RATIO, PROPORTION, PER CENT

Ratio

$$A = \{ \triangle \triangle \triangle \}$$

$$B = \{ \square \square \square \square \square \}$$

Why is it impossible to establish a 1-to-1 matching or a 1-to-1 correspondence between sets A and B ? How would you describe the correspondence of set A to set B shown above? You can express this as follows:

$n(A)$ is to $n(B)$ as 3 is to 5.

Any correspondence between the numbers of two sets is called a **ratio**.

The ratio of 3 to 5 can be denoted by any of the following.

3 to 5 $\frac{3}{5}$ 3:5

Note that fractions can be used to name ratios. A fraction is the preferred form for expressing a ratio.

How would you express the ratio of $n(B)$ to $n(A)$ above?

Oral Answer the questions below about the following sentence.

There are 12 boys and 13 girls in class.

1. What is the ratio of the number of boys to the number of girls?

2. What is the ratio of the number of girls to the number of boys?

Written State the ratio of the number of ★'s to the number of △'s.

- | a | b |
|---|--|
| 1. $\left\{ \begin{array}{c} \star \\ \triangle \triangle \triangle \triangle \end{array} \right\}$ | $\left\{ \begin{array}{c} \star \star \star \\ \triangle \triangle \triangle \triangle \triangle \end{array} \right\}$ |
| 2. $\left\{ \begin{array}{c} \star \star \star \\ \triangle \triangle \triangle \triangle \end{array} \right\}$ | $\left\{ \begin{array}{c} \star \star \star \star \star \\ \triangle \triangle \triangle \end{array} \right\}$ |

Using Ratios



Two different correspondences of the set of squares to the set of stars are shown above. How would you describe each of these correspondences?

Since the two sets are the same in both cases, $\frac{4}{6}$ and $\frac{2}{3}$ must name the same ratio, or $\frac{4}{6} = \frac{2}{3}$. From your study of equivalent fractions, you know that $\frac{4}{6} = \frac{2}{3}$ is a true sentence. Notice that expressing $\frac{4}{6}$ as $\frac{2}{3}$ is equivalent to changing $\frac{4}{6}$ to simplest form.

When expressing a ratio by a fraction, let us agree that the fraction should be in simplest form.

Oral Answer the questions below.

1. If the ratio of the number of boys to that of girls in the orchestra is 3 to 5, do you know how many boys there are in the orchestra?

2. For problem 1, state 3 different possibilities for the number of boys and the number of girls in the orchestra.

Written Write a fraction in simplest form for each ratio below. Do not use mixed numerals.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	8 to 12	$\frac{6}{21}$	13:8
2.	15 to 12	$\frac{18}{66}$	18:81

Write an answer for each of the following problems.

3. The chemical name for sugar is $C_6H_{12}O_6$. In this name C stands for carbon, H stands for hydrogen, and O stands for oxygen. The numerals indicate the number of atoms of each element in one molecule of sugar. What is the ratio of the number of carbon atoms to the number of hydrogen atoms? Of hydrogen atoms to oxygen atoms? Of carbon atoms to oxygen atoms?

4. A team won 6 games and lost 4. What is the ratio of the number of wins to the number of losses? Of losses to wins?

Proportion

The ratio of the number of fiction books to the number of nonfiction books in a school library is 5 to 7. If the library contains 1445 fiction books, how many nonfiction books does it contain?

Let x stand for the number of nonfiction books in the library. What two fractions can you state for the ratio of fiction to nonfiction books?

Since both ratios refer to the same two sets of books, $\frac{5}{7}$ and $\frac{1445}{x}$ stand for the same ratio. This can be stated as a **proportion** as shown below.

$$\frac{5}{7} = \frac{1445}{x}$$

A proportion expresses the equality of two ratios.

Thinking of a proportion as equivalent fractions, you already know how to solve the proportion given above.

$$\begin{aligned}\frac{5}{7} &= \frac{1445}{x} \\ 5x &= 7 \times 1445 \\ x &= \frac{7 \times 1445}{5} \\ x &= 2023\end{aligned}$$

How many nonfiction books does the library contain?

Oral Answer the questions below.

Written Solve each proportion below.

1. What is a proportion?

2. Is $\frac{2}{3} = \frac{9}{12}$ a proportion? Why or why not?

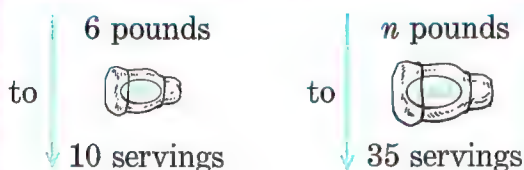
3. How would you solve the proportion $\frac{10}{45} = \frac{n}{36}$? The proportion $\frac{56}{a} = \frac{7}{3}$? The proportion $\frac{n}{15} = \frac{64}{12}$?

	a	b	c
1.	$\frac{3}{7} = \frac{a}{56}$	$\frac{1}{6} = \frac{16}{r}$	$\frac{n}{7} = \frac{27}{63}$
2.	$\frac{x}{6} = \frac{20}{8}$	$\frac{15}{a} = \frac{9}{24}$	$\frac{11}{24} = \frac{c}{72}$
3.	$\frac{n}{15} = \frac{64}{12}$	$\frac{26}{16} = \frac{n}{8}$	$\frac{72}{126} = \frac{r}{14}$
4.	$\frac{7}{13} = \frac{14}{x}$	$\frac{15}{a} = \frac{35}{28}$	$\frac{n}{120} = \frac{15}{18}$

Problem Solving

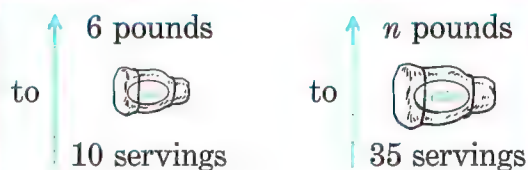
If 6 pounds of meat are needed for 10 servings, how many pounds of meat are needed for 35 servings?

There are several acceptable proportions for this problem. In each of the four proportions explained below, n stands for the number of pounds of meat needed for 35 servings.



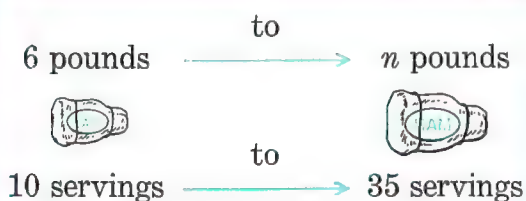
The ratios may be obtained by comparing the number of pounds to the number of servings.

$$\frac{6}{10} = \frac{n}{35}$$



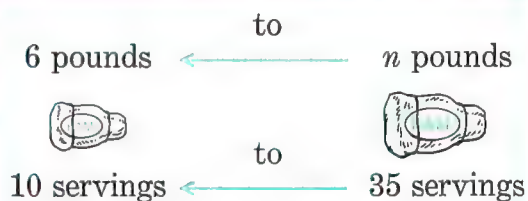
The ratios may be obtained by comparing the number of servings to the number of pounds.

$$\frac{10}{6} = \frac{35}{n}$$



The ratios may be obtained by comparing the first amount of meat to the second and by comparing the first number of servings to the second.

$$\frac{6}{n} = \frac{10}{35}$$



The ratios may be obtained by comparing the second amount of meat to the first and by comparing the second number of servings to the first.

$$\frac{n}{6} = \frac{35}{10}$$

Show that each proportion above is equivalent to $10n = 210$.
 How many pounds of meat are needed for 35 servings?

Oral Tell how you could think about the following problem to obtain each proportion below.

A machine can produce 14 toys in 10 minutes. At this rate, how many toys can it produce in 21 minutes?

a

b

$$1. \quad \frac{10}{14} = \frac{x}{21}$$

$$\frac{10}{x} = \frac{14}{21}$$

$$2. \quad \frac{x}{10} = \frac{21}{14}$$

$$\frac{14}{10} = \frac{21}{x}$$

Written Write a proportion for each of the following problems. Solve the proportion. Write an answer for the problem.

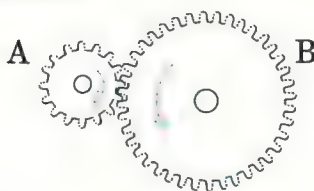
1. For his outboard motor, Mr. Jones mixes 2 quarts of oil with 5 gallons of gasoline. How many quarts of oil does he need for 15 gallons of gasoline?

2. A contractor mixes 2 parts of cement with 5 parts of sand in making concrete. He used 700 pounds of cement on a certain project. How many pounds of sand were needed for that project?

3. If a dozen oranges cost 42 cents, what should 10 oranges cost?

4. In a certain town the ratio of the number of cars to that of trucks is 34 to 7. If there are 28 trucks in that town, how many cars are there?

5. When gear A below makes 15 revolutions gear B makes 6 revolutions. How many revolutions will gear B make while gear A makes 35 revolutions?



6. Under normal driving conditions a car uses 4 gallons of gasoline while traveling 68 miles. How far can you expect to drive this car on 18 gallons of gasoline?

7. Of the two candidates in an election, Mr. Brown is favored 5 to 4 over Mr. Collins. Mr. Brown receives 3525 votes. How many votes can Mr. Collins expect?

Can you do this? Solve each problem below.

1. One number is 3 less than another number and the ratio of the first number to the second number is 3 to 4. What are the two numbers?

2. The ratio of the number of boys to that of girls in the band is 6 to 7. If there are 39 pupils in the band, how many are boys? How many are girls?

3. The ratio of $a+7$ to $a-3$ is 3 to 1. What whole number does a stand for?

Per Cent

The symbol % at the right means *per hundred*. How many voters out of every 100 voted Democratic? Voted Republican? Voted for an Independent candidate? How do you know that all votes are accounted for?

Election Results

DEMOCRATS	43%
REPUBLICANS	43%
INDEPENDENTS	14%

The symbol 43% is read *forty-three per cent*. It means 43 out of 100 or the ratio of 43 to 100 or $\frac{43}{100}$.

If x stands for a number, then $x\%$ expresses the ratio of x to 100 or $\frac{x}{100}$.

The expressions 7%, 100%, and 125% are defined below.

$$7\% = \frac{7}{100} \text{ or } .07$$

$$100\% = \frac{100}{100} \text{ or } 1$$

$$125\% = \frac{125}{100} = \frac{5}{4} \text{ or } 1\frac{1}{4} \text{ or } 1.25$$

Oral Answer the questions below.

1. For what replacements of x will $x\%$ name a number less than 1?
2. For what replacements of x will $x\%$ name a number greater than 1?

Read each symbol below. Tell whether it names a number less than one, equal to one, or greater than one.

	a	b	c
3.	3%	52%	108%
4.	17%	97%	236%

Written Copy. Complete the table so that all the symbols in each row name the same number. Express all fractions in simplest form.

	Per cent	Fraction	Decimal
1.	17%	_____	_____
2.	_____	$\frac{43}{100}$	_____
3.	_____	_____	.09
4.	84%	_____	_____
5.	_____	_____	1.37
6.	_____	$\frac{3}{4}$	_____
7.	425%	_____	_____

Changing Per Cents to Decimals

$$247\% = \frac{247}{100} = \frac{200+47}{100} = \frac{200}{100} + \frac{47}{100} = 2.47$$

$$24.7\% = \frac{24.7}{100} = \frac{24.7}{100} \times \frac{10}{10} = \frac{247}{1000} = .247$$

$$2.47\% = \frac{2.47}{100} = \frac{2.47}{100} \times \frac{100}{100} = \frac{247}{10000} = .0247$$

$$.247\% = \frac{.247}{100} = \frac{.247}{100} \times \frac{1000}{1000} = \frac{247}{100000} = .00247$$

In each example above, compare the placement of the decimal point in the per cent notation with that in the equivalent decimal. What pattern can you discover? Does this pattern hold for the following cases also?

$$207\frac{1}{2}\% = \frac{207\frac{1}{2}}{100} = \frac{200+7\frac{1}{2}}{100} = \frac{200}{100} + \frac{7\frac{1}{2}}{100} = 2.07\frac{1}{2}$$

$$20.7\frac{1}{2}\% = \frac{20.7\frac{1}{2}}{100} = \frac{20.7\frac{1}{2}}{100} \times \frac{10}{10} = \frac{207\frac{1}{2}}{1000} = .207\frac{1}{2}$$

Oral Express each of the following per cents as decimals.

Written Write a decimal for each per cent below.

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|----------|----------|----------|
| 1. | 28% | 2.8% | .28% |
| 2. | 352% | 35.2% | 3.52% |

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|----------|----------|----------|
| 1. | 71% | .71% | 710% |
| 2. | 5.3% | 53% | .053% |

Answer the following questions.

Write a per cent for each decimal below.

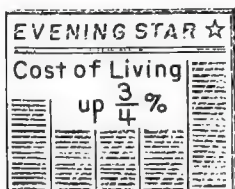
3. What pattern can you use to change a per cent to a decimal?

4. Express 3.71 in per cent notation.

5. What pattern can you use to change a decimal to a per cent?

- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|----------|----------|----------|
| 3. | .92 | 9.2 | .092 |
| 4. | 3.17 | .317 | .0317 |
| 5. | 48.1 | 4.81 | .481 |

Fractional Per Cents



To change $\frac{3}{4}\%$ to a fraction, you can proceed in either way shown below.

$$\frac{3}{4}\% = \frac{3}{4} \times \frac{1}{100} = \frac{3}{400}$$

$$\frac{3}{4}\% = .75\% = .0075 = \frac{75}{10000} = \frac{3}{400}$$

Consider changing $\frac{2}{7}\%$ to a fraction by using both of these ways. What difficulty arises when you attempt to change $\frac{2}{7}\%$ to a fraction by using decimals? For what fractions would this difficulty occur? To avoid such difficulties, let us interpret the $\%$ notation as shown below.

$$4\frac{1}{2}\% = 4\frac{1}{2} \times \frac{1}{100} = \frac{9}{2} \times \frac{1}{100} = \frac{9}{200}$$

$$33\frac{1}{3}\% = 33\frac{1}{3} \times \frac{1}{100} = \frac{100}{3} \times \frac{1}{100} = \frac{1}{3}$$

This interpretation of the $\%$ notation leads to an easy way of changing fractional per cents to fractions.

$$7\frac{1}{7}\% = 7\frac{1}{7} \times \frac{1}{100} = \frac{50}{7} \times \frac{1}{100} = \frac{50}{700} = \frac{1}{14}$$

$$.5\frac{1}{4}\% = .5\frac{1}{4} \times \frac{1}{100} = \left(5\frac{1}{4} \times \frac{1}{10}\right) \times \frac{1}{100} = \frac{21}{4} \times \frac{1}{10} \times \frac{1}{100} = \frac{21}{4000}$$

$$.06\frac{2}{3}\% = .06\frac{2}{3} \times \frac{1}{100} = \left(6\frac{2}{3} \times \frac{1}{100}\right) \times \frac{1}{100} = \frac{20}{3} \times \frac{1}{100} \times \frac{1}{100} = \frac{20}{30000} = \frac{1}{1500}$$

Observe the per cent and the fraction connected by the blue lines above. What simple pattern can you discover for changing a fractional per cent to a fraction?

Oral Read each numeral below.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	$5\frac{2}{3}\%$	$\frac{4}{5}\%$.4%	110%
2.	$3\frac{1}{8}\%$	$3\frac{1}{2}\%$.25%	160%
3.	$\frac{5}{7}\%$	$7\frac{1}{4}\%$	1.2%	250%
4.	$2\frac{1}{6}\%$	$12\frac{1}{3}\%$	7.8%	460%

Tell the decimal equivalent of each per cent below.

	<i>a</i>	<i>b</i>	<i>c</i>
5.	23%	4%	$5\frac{1}{4}\%$
6.	2%	$7\frac{1}{2}\%$	1.3%
7.	$8\frac{1}{5}\%$.6%	13%
8.	$5\frac{3}{4}\%$	15%	130%
9.	$3\frac{4}{5}\%$	17.2%	643%

Tell how you would change each per cent below to a fraction in simplest form.

	<i>a</i>	<i>b</i>	<i>c</i>
10.	$\frac{1}{2}\%$	$2\frac{1}{4}\%$	$\frac{4}{5}\%$
11.	$5\frac{2}{3}\%$	$\frac{3}{8}\%$	120%
12.	$\frac{5}{7}\%$	15%	$2\frac{1}{6}\%$
13.	$6\frac{1}{3}\%$	5%	4%
14.	$8\frac{5}{8}\%$	$.04\frac{3}{8}\%$	$.8\frac{2}{7}\%$

Written Copy. Complete the table so that all numerals in each row name the same number. Express all fractions in simplest form.

	<i>Fraction</i>	<i>Decimal</i>	<i>Per cent</i>
1.	$\frac{1}{4}$.25	25%
2.	$\frac{1}{40}$	_____	_____
3.	_____	_____	$.37\frac{1}{2}\%$
4.	_____	.64	_____
5.	_____	_____	65%
6.	$1\frac{7}{20}$	_____	_____
7.	_____	3.08	_____
8.	_____	_____	$\frac{5}{8}\%$
9.	$\frac{1}{15}$	_____	_____
10.	_____	$.003\frac{1}{2}$	_____
11.	_____	_____	$8\frac{1}{3}\%$
12.	$17\frac{1}{50}$	_____	_____
13.	_____	_____	110%
14.	$\frac{4}{25}$	_____	_____
15.	_____	2.75	_____
16.	$13\frac{4}{5}$	_____	_____
17.	_____	$.2\frac{6}{7}$	_____
18.	_____	_____	.008%

Can you do this? Arrange the following from least to greatest.

- $7\frac{1}{2}\%$, $.7\frac{1}{2}$, $\frac{5}{80}$, .0725
- .275, $\frac{11}{4}$, $2\frac{3}{4}\%$, $\frac{1}{40}$

Per Cent of a Number

Mr. Waters paid a 4% sales tax on a \$75 purchase. How much sales tax did he pay?

Since you need to find 4% of 75, you can translate the problem into the open sentence below and solve it as shown.

$$\begin{aligned} a &= 4\% \times 75 \\ &= \frac{\cancel{4}^1}{\cancel{100}^{25}} \times \cancel{75}^3 \\ &= 3 \end{aligned}$$

When the per cent expression has a terminating decimal equivalent, as it does above, it is also possible to compute the result as follows.

$$\begin{array}{r} 75 \\ \times .04 \\ \hline 3.00 \text{ or } 3 \end{array}$$

How much sales tax did he pay?

Study how the problem below is solved.

Mr. Carder has a 240-acre farm of which $12\frac{1}{2}\%$ is covered with water. How many acres are covered with water?

$$12\frac{1}{2}\% \times 240 = \frac{\cancel{25}^1}{\cancel{200}^{20}} \times \cancel{240}^{30} = 30$$

How many acres are covered with water? Could this problem be solved by using decimals? If so, how would you solve it?

Explain each computation below.

$$\begin{aligned} \frac{5}{6}\% \times 24 &= \frac{\cancel{5}^1}{\cancel{6}^2} \times \frac{1}{\cancel{100}^{20}} \times \cancel{24}^1 = \frac{1}{5} \\ 8\frac{1}{3}\% \times 120 &= \frac{\cancel{25}^1}{\cancel{3}^1} \times \frac{1}{\cancel{100}^{10}} \times \cancel{120}^{10} = 10 \end{aligned}$$

Oral Tell how you would solve each open sentence below.

a

b

1. $20\% \times 35 = x$ $17\% \times 3429 = n$

2. $37\frac{1}{2}\% \times 32 = m$ $33\frac{1}{3}\% \times 57 = r$

3. $75\% \times 16 = t$ $323\% \times 7021 = s$

Written Copy. Solve each open sentence.

a

b

1. $5\% \times 40 = n$ $10\% \times 120 = n$

2. $4\% \times 92 = n$ $75\% \times 180 = n$

3. $30\% \times 90 = n$ $73\% \times 18 = n$

4. $7\% \times 2.4 = n$ $125\% \times 36 = n$

5. $12\frac{1}{2}\% \times 72 = n$ $250\% \times 22 = n$

6. $8\frac{1}{3}\% \times 48 = n$ $80\% \times 9.1 = n$

7. $23\% \times 42 = n$ $\frac{3}{4}\% \times 12 = n$

8. $33\frac{1}{3}\% \times 42 = n$ $5.2\% \times 18 = n$

9. $.7\% \times 3.2 = n$ $.15\% \times 8 = n$

10. $\frac{2}{3}\% \times 24 = n$ $400\% \times 13 = n$

11. $\frac{3}{8}\% \times 48 = n$ $100\% \times 75 = n$

12. $47\% \times 28 = n$ $6\% \times 5428 = n$

13. $\frac{3}{5}\% \times \frac{3}{4} = n$ $7.8\% \times 15 = n$

Solve each problem below.

14. A certain new car sold for \$2450 and was subject to a 3% sales tax. How much sales tax would be paid on this car?

15. A salesman receives a commission of 8% on his total sales. During May his total sales amounted to \$21,550. How much commission did he earn during May?

16. A chemical was advertised as containing less than $\frac{3}{8}\%$ impurities by weight. If the ad is true, what is the maximum weight of the impurities permissible in 1000 pounds of the chemical?

17. There are 20,936 eligible voters in Centerville. In a recent election in Centerville, 62.5% of the eligible voters voted. How many people voted?

Can you do this? Answer each question below.

1. A man reduces his business by 100%. What is that man doing?

2. A man increases his business by 100%. What is that man doing?

3. A brick weighs 2 pounds plus 50% of its total weight. How many pounds will 2 bricks weigh?

4. Is $80\% \times \frac{1}{2}$ equal to $80 \times \frac{1}{2}\%$?

Proportion and Per Cent

Since per cent indicates the ratio of some number to 100, you can use a proportion to solve any per cent problem.

A man earned \$1248 and saved \$312 of it. What per cent of his earnings did he save?

To solve the problem you must answer the question,

\$312 is what per cent of \$1248?

Or you may think: 312 is to 1248 as some number x is to 100.

$$\frac{312}{1248} = \frac{x}{100} \quad \leftarrow \begin{array}{l} x \text{ stands for a number.} \\ \frac{x}{100} \text{ stands for a per cent.} \end{array}$$

$$1248x = 312 \times 100$$

$$x = \frac{\overset{1}{\cancel{312}} \times \overset{25}{\cancel{100}}}{\underset{+}{\cancel{1248}}}$$

$$x = 25$$

He saved $\frac{25}{100}$ or 25% of his earnings.

John answered 28 test questions correctly. This was 80% of the total number of questions on the test. How many questions were there on the test?

To solve this problem, what question must you answer? What proportion could you use?

$$\frac{28}{n} = \frac{80}{100} \quad \text{What does } n \text{ stand for?}$$

$$\frac{28}{n} = \frac{4}{5} \quad \text{How is } \frac{4}{5} \text{ obtained?}$$

$$4n = 5 \times 28$$

$$n = \frac{\overset{7}{\cancel{5} \times \cancel{28}}}{\underset{1}{\cancel{4}}}$$

$$n = 35$$

How many questions were there on the test?

Oral Tell which of the following proportions can be used to answer each question below.

a. $\frac{n}{36} = \frac{25}{100}$

d. $\frac{36}{n} = \frac{25}{100}$

b. $\frac{n}{9} = \frac{36}{100}$

e. $\frac{9}{36} = \frac{n}{100}$

c. $\frac{9}{n} = \frac{25}{100}$

f. $\frac{36}{9} = \frac{n}{100}$

1. What number is 25% of 36?
2. 9 is what per cent of 36?
3. 9 is 25% of what number?
4. 36 is what per cent of 9?
5. What number is 36% of 9?
6. 36 is 25% of what number?

Written Solve each open sentence below.

a

b

1. $45\% \times 72 = n$ $x\% \times 75 = 30$

2. $25\% \times n = 40$ $d \times 20\% = 15$

3. $40\% \times \frac{3}{4} = c$ $10 = 125\% \times r$

4. $\frac{2}{5}\% \times \frac{5}{7} = m$ $.5\% \times 240 = t$

Solve the following problems.

5. What number is 20% of 30?
6. 8 is what per cent of 32?
7. 9 is 30% of what number?
8. What number is $\frac{3}{4}\%$ of 36?

9. What number is 140% of 15?

10. 2.7 is what per cent of 36?

11. 75 is 125% of what number?

12. In a class of 25 pupils, 23 were present. What per cent of the pupils were absent?

13. One month the Mullens saved \$80. This was 16% of the total family income for the month. How much was the family income for the entire month?

14. Bill received 4% interest per year on his savings of \$240. What was the amount of interest in one year?

15. If a person earns \$120 and saves \$24 of his earnings, what per cent of his earnings does he save?

Can you do this? Each of two men buy a portable TV set for \$100. Because of a sale, there is a 10% discount. There is also a 10% discount for paying cash. One man takes a straight 20% discount. The other man takes 10% off the list price, and then takes 10% off the reduced price. Who got the better deal and by how much?

Tell why If you increase your \$80 income by 10% and later decrease the new income by 10%, your new income will be less than \$80.

Solving Problems

Write an open sentence for each problem. Solve the open sentence. Answer the problem.

1. In a flower decoration the ratio of the numbers of white lilacs to purple lilacs is 4 to 5. What per cent of the flowers are white?

2. Jim sold 6 kittens. This was 75% of his kittens. How many kittens did he have before selling any?

3. In an elementary school with 360 pupils enrolled, 55% of the pupils were girls. How many girls were enrolled in the school?

4. Mr. Bock paid \$3,600 for a lot. He built an \$18,000 house on it. The price of the lot was what per cent of the cost of the house?

5. In a high school 7% of the pupils were absent one day. There were 98 pupils absent that day. How many pupils were enrolled in that high school?

6. A punch bowl contains 40 cups of punch. If 60% of the punch is fruit juice and the rest carbonated water, how many cups of fruit juice and of water are there in the punch bowl?

7. Mr. Lucas saved \$42 one week. This was 6% of his monthly salary. How much was his monthly salary?

8. A football player completed 15 out of 25 passes in one game. What per cent of his passes did he complete?

9. On a spelling test, Mary had 18 out of 20 words spelled correctly. What per cent of the words did Mary spell correctly?

10. On a mathematics test Bill worked 23 problems correctly. This was 92% of the problems. How many problems were there on the test?

11. During vacation, John saved 4 out of every 5 dollars that he earned. What per cent of his earnings did John save?

12. A basketball team played 25 games one season. It won 80% of its games. How many games did the team win?

Can you do this? Answer the following questions.

1. A baseball team won 16 games and lost 9. What per cent of its games did the team win? What per cent of the games did the team lose? (Hint: Number of games played is $16+9$.)

2. Lew won 6 out of 8 games. Dick won 8 out of 10 games. Who has the better record? (Hint: Compare the per cent of games won by Lew to the per cent of games won by Dick.)

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Every ratio can be expressed by a fraction. (191)
2. A proportion is a sentence that states that two ratios are equal. (193)
3. A problem may be translated into more than one proportion. (194)
4. A per cent expresses a ratio of some number to 100. (196)
5. Either fractions or decimals can be used to solve problems involving per cent. (200)
6. A proportion can be used to solve any per cent problem. (202)

Words to Know

1. Ratio (191)
2. Proportion (193)
3. Per cent (196)

Questions to Discuss

1. What does it mean to say that the ratio of the number of one set to the number of another set is $\frac{2}{5}$? (191)
2. How would you solve the proportion $\frac{3}{8} = \frac{x}{72}$? (193)
3. How do you read 58%? What does it mean? (196)

4. How is the placement of the decimal point changed when a decimal is changed to a per cent? (197)

5. How would you change $\frac{3}{4}\%$ to a fraction? (198)

6. How would you find 5% of 18? (200)

7. How would you express "15 is what per cent of 75" as a proportion? (202)

Written Practice

Solve each proportion below. (193)

a	b	c
1. $\frac{5}{8} = \frac{n}{72}$	$\frac{13}{7} = \frac{117}{a}$	$\frac{c}{6} = \frac{65}{78}$
2. $\frac{15}{x} = \frac{45}{24}$	$\frac{5}{12} = \frac{x}{84}$	$\frac{7}{n} = \frac{98}{126}$

Complete the table below so that all the symbols in each row name the same number. (199)

	Per cent	Fraction	Decimal
3.	—	$\frac{17}{25}$	—
4.	$12\frac{1}{2}\%$	—	—

Solve each open sentence. (200)

a	b
5. $28\% \times 45 = n$	$a\% \times 84 = 21$
6. $\frac{4}{5}\% \times \frac{3}{32} = x$	$.7\% \times 17 = r$

Self-Evaluation

Part 1 Write an answer for each of the following questions.

1. On the American flag, what is the ratio of the numbers of stars to stripes? Of stripes to stars?

2. A team won 8 games and lost 6 games. What is the ratio of the numbers of wins to losses? Of losses to wins? Of wins to the total number of games played?

3. A box contained 5 red marbles, 15 blue marbles, and 5 black marbles. What is the ratio of the numbers of blue marbles to black marbles? Of red marbles to black marbles?

Part 2 Solve each of the following proportions.

a	b	c
1. $\frac{3}{16} = \frac{6}{n}$	$\frac{2}{6} = \frac{x}{9}$	$\frac{5}{23} = \frac{15}{c}$
2. $\frac{17}{a} = \frac{51}{39}$	$\frac{x}{20} = \frac{21}{28}$	$\frac{36}{21} = \frac{x}{14}$
3. $\frac{27}{31} = \frac{81}{n}$	$\frac{a}{7} = \frac{36}{42}$	$\frac{3}{x} = \frac{36}{132}$

Solve each problem below.

4. A gardener mixes 3 pints of insecticide with 10 pints of water to make a plant spray. How much water is needed for 30 pints of insecticide?

5. If a dozen eggs cost 57 cents, what should 8 eggs cost?

Part 3 Copy. Complete the table so that all numerals in a row name the same number.

	<i>Fraction</i>	<i>Decimal</i>	<i>Per cent</i>
1.	$\frac{3}{20}$	_____	_____
2.	_____	_____	$\frac{2}{5}\%$
3.	$\frac{1}{3}$	_____	_____
4.	_____	_____	37.5%
5.	_____	3.17	_____

Part 4 Solve each open sentence below.

a	b
1. $35\% \times 24 = x$	$28 \times n\% = 35$
2. $36 \times \frac{3}{4}\% = m$	$144 = 180\% \times k$
3. $3.2\% \times 85 = y$	$\frac{1}{5}\% \times \frac{25}{32} = x$

Solve each problem below.

4. Arthur has 36 model airplanes. He built 27 of them. What per cent of his model airplanes did he build?

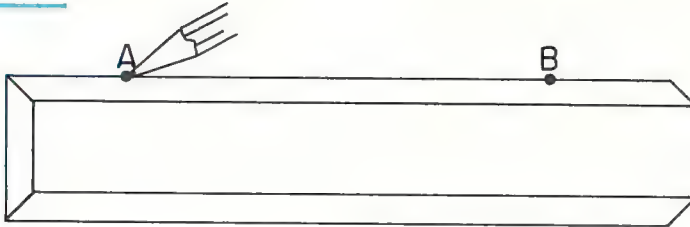
5. By weight the human body is about 60% water. John said that the water in his body weighed 65 pounds. How much did John weigh?

6. The Eagles won $66\frac{2}{3}\%$ of the games they played. If they won 36 games, how many games did they play?

Chapter 9

LINE, CIRCLES, AND ANGLES

Points and Lines



The dots shown above are pictures of **points** in the same way that numerals are names for numbers. We usually name points with capital letters, like *point A* or *point B*.

By drawing along the straightedge above, you draw a picture of a **line** which *passes through* points A and B. Points A and B are *on* the line.



Line AB or \overleftrightarrow{AB}

\overleftrightarrow{AB} is read *line AB*. When we say *line* we mean *straight line*. A line has no endpoints. It extends indefinitely in both of its directions. How is this indicated above?

Oral Answer the questions below.

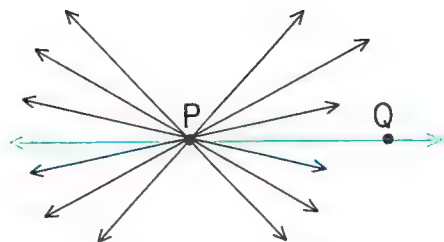
1. What do the arrows in the picture of \overleftrightarrow{AB} above indicate?
2. How do you read \overleftrightarrow{BA} ? Does it name the same line as \overleftrightarrow{AB} ?
3. Could you locate more points on \overleftrightarrow{AB} above? How many more?

Written Draw the following.

1. Point A on line BC
2. Point T not on \overleftrightarrow{BC}
3. A line through a given point N
4. A line not through point N
5. Two lines through point K

Sets of Points

How many lines are drawn through point P at the right? Could more lines be drawn through point P? How many lines can be drawn through both point P and point Q?



Think of points P and Q and all points on \overline{PQ} that are between P and Q. This set of points is called a **line segment**.



Line segment PQ or \overline{PQ}

Line segment QP or \overline{QP}

Points P and Q are called **endpoints** of \overline{PQ} . The endpoints can be named in either order, \overline{QP} or \overline{PQ} , to name a line segment.

Now think of point P and all the points on \overline{PQ} that are on the same side of P as Q. This set of points is called a **ray**.

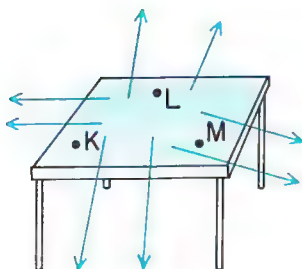


Ray PQ or \overrightarrow{PQ}

Point P is called *the* endpoint of \overrightarrow{PQ} . The endpoint is named first when naming a ray. What ray is shown below?



\overrightarrow{QP}



Think of all the points on the table top at the left and its extension. Such a set of points is called a **plane**. A plane extends indefinitely in all of its directions but has no thickness. To name *plane* KLM, name any 3 points not on the same line in the plane.

Oral Tell how many endpoints each of the following geometric figures has. Tell also which point(s) would be the endpoint(s).

- | | a | b | c |
|----|-----------------|-----------------|-----------------|
| 1. | \overline{RS} | \overline{DE} | \overline{GF} |
| 2. | \overline{XY} | \overline{CD} | \overline{TV} |

Tell which set of points is named by each of the following.

- | | a | b | c |
|----|-----------------|-----------------|-----------|
| 3. | \overline{ST} | \overline{ST} | Point R |
| 4. | \overline{ST} | \overline{TS} | Plane ABC |

Written Make a drawing to illustrate each of the following.

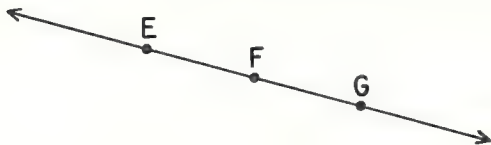
- | | a | b | c | d |
|----|-----------------|-----------------|-----------------|-----------------|
| 1. | \overline{DE} | \overline{DE} | \overline{DE} | \overline{ED} |

Write the names requested in each of the following.

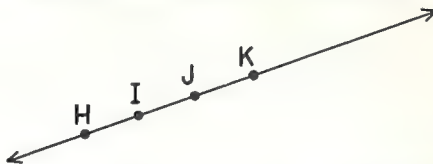
2. Name all line segments and all rays in the figure below.



3. Name 3 different line segments and 6 different rays in this figure.



4. Name 6 different line segments and 8 different rays in this figure.

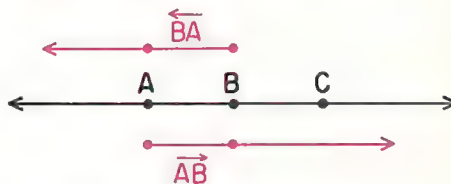


5. Name 10 different line segments and 10 different rays in this figure.



Tell how Notice the pattern for the number of line segments and the pattern for the number of rays in *Written 2-5*. Without a drawing, how can you tell the number of line segments and the number of rays when 6 points are marked on a line?

Can you do this? The union of \overline{AB} and \overline{BA} , denoted by $\overline{AB} \cup \overline{BA}$, is the set of points on \overline{AB} or \overline{BA} or both.

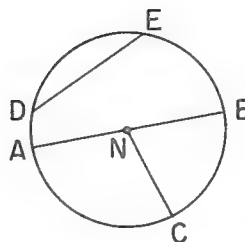
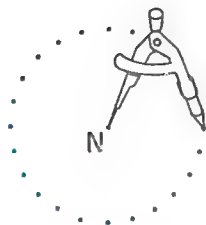


Therefore, $\overline{AB} \cup \overline{BA} = \overline{AB}$.

Use \overline{AC} above to write a single name, like AB or \overline{AB} , for each of the following.

- | | a | b |
|----|------------------------------------|------------------------------------|
| 1. | $\overline{AB} \cup \overline{AB}$ | $\overline{BA} \cup \overline{AB}$ |
| 2. | $\overline{AB} \cup \overline{BC}$ | $\overline{AC} \cup \overline{BA}$ |

Circles



By placing the steel tip of a compass at point N, and without changing the opening of the compass, you can locate many points that are the same distance from point N. The complete set of such points is called a **circle**.

A *circle* is a set of points in a plane such that each point is the same distance from some given point. The given point is called the *center* of the circle.

Since a circle has only one center, you can refer to the circle by referring to its center. In this way, you can refer to the circle above as circle N.

A line segment joining the center of a circle with any point on the circle is called a **radius** of that circle.

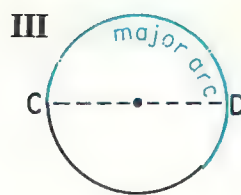
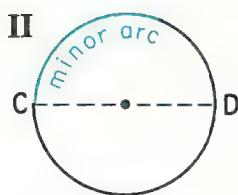
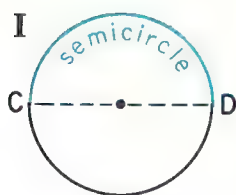
\overline{NC} is a radius of circle N above. Name two more radii (*rā'dē ī'*, plural of radius) of circle N. How many radii does a circle have?

A line segment joining any two points on a circle is called a **chord** of that circle.

\overline{DE} is a chord of circle N. Name another chord of circle N. How many chords does a circle have? Could any chord of circle N be longer than \overline{AB} ? Does \overline{AB} pass through the center of circle N?

A chord that passes through the center of a circle is called a **diameter** of that circle.

\overline{AB} is a diameter of circle N. How many diameters does a circle have?



In each drawing above, points C and D separate the circle into two parts. Each part is called an **arc**. If the endpoints of an arc are also endpoints of a diameter, as in drawing I, the arc is called a **semicircle**.

In drawings II and III one endpoint of the diameter is also an endpoint of the arc shown in color. Is the other endpoint of this diameter *on* the colored arc in drawing II? In drawing III? What reason can you give for calling the colored arc in II a **minor arc**? The colored arc in III a **major arc**? How is a major arc different from a semicircle?

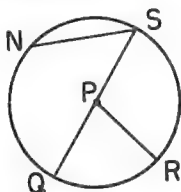
Let us agree that, unless otherwise specified, such names as arc CD will always refer to the minor arc.

Oral Tell which set of points in the figure below is named by each of the following expressions.

1. Radius PR

2. Diameter QS

3. Chord SN



2. Mark point C on circle B so that it is inside circle T.

3. Draw \overline{RC} . Is \overline{RC} a radius or a chord of circle B?

4. Is \overline{TB} a radius, a chord, or a diameter of circle B? Of circle T?

Written Draw a circle with point T as its center. Mark any point B on the circle. Use B as the center and \overline{TB} as a radius and draw another circle. Now do the following.

1. Mark point R on circle B so that it is outside circle T.

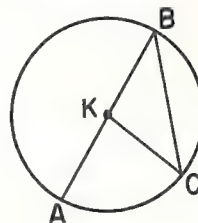
Use circle K below to name the following figures.

5. Three radii

6. Two chords

7. Three arcs

8. A diameter



Congruent Line Segments

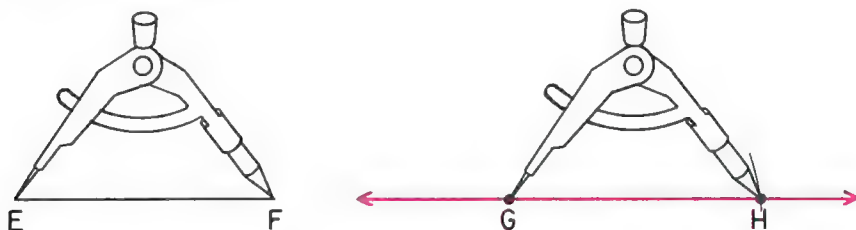


To compare the lengths of the line segments above, lay a strip of paper along \overline{AB} and make marks at its endpoints as shown. Now place the paper strip along \overline{CD} to see if the marks fit with points C and D. If they do, you may conclude that the two different line segments have the same length.

We cannot very well say that $\overline{AB} = \overline{CD}$ since they are different line segments. Instead, we say that \overline{AB} is *congruent to* \overline{CD} or that \overline{CD} is *congruent to* \overline{AB} , which can be abbreviated as $\overline{AB} \cong \overline{CD}$ or $\overline{CD} \cong \overline{AB}$.

If two line segments have the same length, they are called **congruent** line segments.

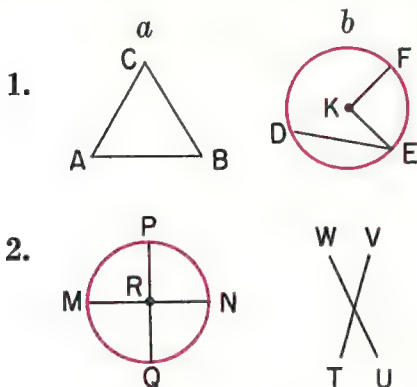
A compass is convenient for constructing a line segment congruent to a given line segment. Let \overline{EF} below represent the given line segment.



On your paper, draw a line as shown in red above. Mark some point G on that line. Now adjust your compass as shown in the first figure. Transfer this compass opening onto the line by placing the tip of the compass at point G and drawing an arc that intersects the line. Call that point of intersection H. By this construction, $\overline{EF} \cong \overline{GH}$.

It is understood for all geometric construction that only a straightedge and a compass will be used. A ruler is used only as a straightedge and never as a graduated measuring scale.

Oral Use a strip of paper to decide which line segments are congruent in each figure below.



Answer the following questions.

3. How would you read $\overline{AC} \cong \overline{BC}$?

4. What does $\overline{AC} \cong \overline{BC}$ mean?

5. Do two line segments have to point in the same direction in order to be congruent? (See *Oral* 1–2.)

6. Can two congruent line segments have a point in common?

7. Can two congruent line segments intersect each other?

8. In *Oral* 2a, is it correct to say that $\overline{MN} \cong \overline{MN}$? What does this mean? Is every line segment congruent to itself?

Written Make the following constructions on your paper.

1. Draw a horizontal line segment about 2 inches long. Construct a

vertical line segment that is congruent to it.

2. Draw a vertical line segment about 3 inches long. Construct a horizontal line segment that is congruent to it.

3. Draw a line segment that is neither horizontal nor vertical. Construct a horizontal line segment that is congruent to it.

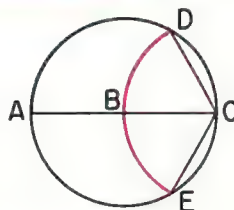
Write a reason for the truth of each statement below.

4. All radii of a circle are congruent line segments.

5. All diameters of a circle are congruent line segments.

6. A radius of a circle is not congruent to a diameter of that circle.

Can you do this? Study this figure.

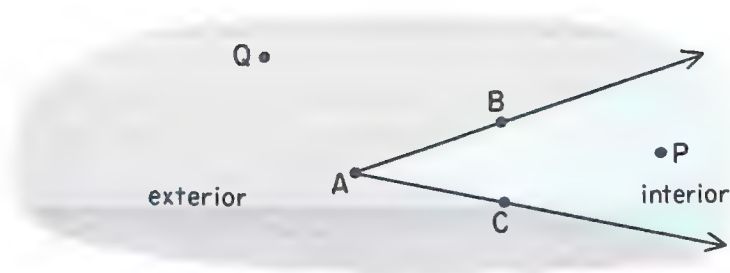


1. Is $\overline{DC} \cong \overline{CE}$? Why?

2. Does it seem reasonable that $\text{arc } DC \cong \text{arc } CE$?

3. How would you define congruent arcs?

Angles



Rays AB and AC start at the same point. Since each ray is a set of points, we can think of the above figure as the union of these two rays. The figure is called an **angle**.

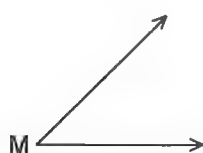
An *angle* is the union of two rays having a common endpoint. The common endpoint is called the **vertex** of the angle and the two rays are called **sides** of the angle.

The angle above can be named “angle BAC” or “angle CAB.” Notice that the letter naming the vertex is the middle letter. These names can be abbreviated as

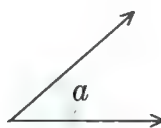
$$\angle BAC \quad \text{and} \quad \angle CAB.$$

The *interior* and the *exterior* of $\angle BAC$ are shown above. Point P is in the interior of $\angle BAC$; point Q is in the exterior of $\angle BAC$; and points A, B, and C are *on* $\angle BAC$.

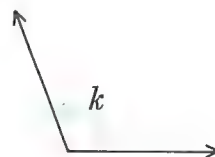
It is often convenient to use a shorter symbol than $\angle BAC$ for naming an angle. If no confusion is possible, the vertex letter alone can be used to name an angle, as for $\angle M$ below. Angles can also be named by a small letter together with an arc as shown below for $\angle a$ and $\angle k$.



angle M or $\angle M$

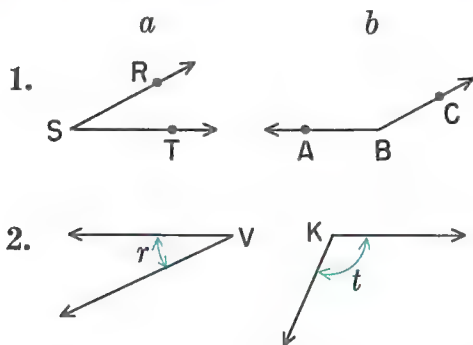


angle a or $\angle a$



angle k or $\angle k$

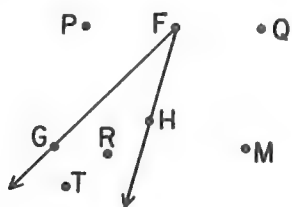
Oral Name each angle below in two different ways.



As you read each expression below, tell which point is the vertex of the angle and which rays are the sides of the angle.

- | | a | b | c |
|----|--------------|--------------|--------------|
| 3. | $\angle RST$ | $\angle TSR$ | $\angle TRS$ |
| 4. | $\angle EFG$ | $\angle KQM$ | $\angle QKM$ |

Use the following figure to answer Oral 5–9.



5. Name 3 points in the exterior of $\angle GFH$.

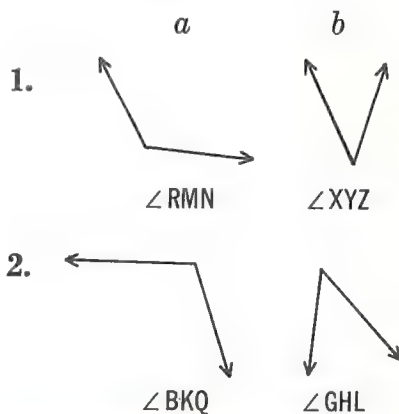
6. If a line segment joins 2 points in the exterior of an angle, in how many points does it intersect the angle?

7. Name two points in the interior of $\angle GFH$.

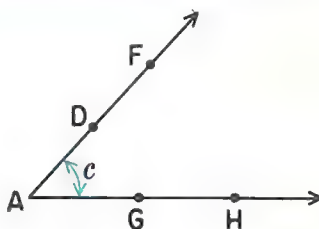
8. If a line segment joins 2 points in the interior of an angle, in how many points does it intersect the angle?

9. If a line segment joins a point in the interior and a point in the exterior of an angle, in how many points does it intersect the angle?

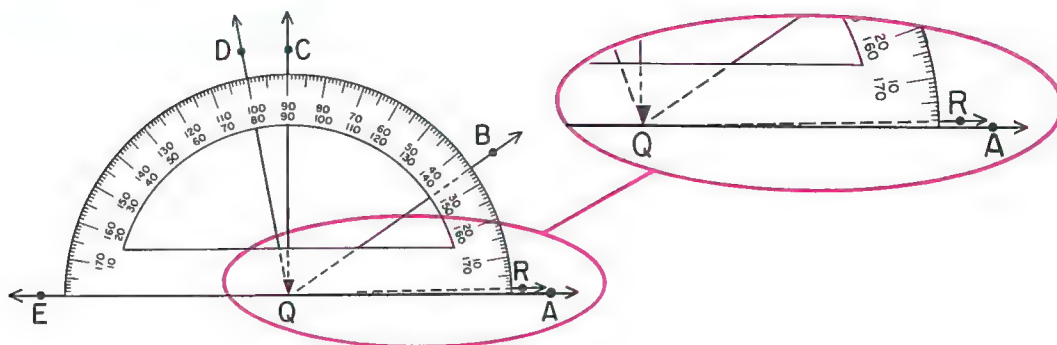
Written Copy each drawing below. Label each drawing so that the expression below it is a name for the angle.



Can you do this? Name the angle below in ten different ways.



Measures of Angles



A protractor, as shown above, is a model of a *semicircle* and the diameter joining its endpoints. The semicircle is separated into 180 arcs of the same size. The center of the circle is marked, as point Q above.

Notice that the sides of $\angle AQR$ pass through the endpoints of one of the small arcs. The measurement of $\angle AQR$ is 1 **degree** (1°), or the **degree-measure** of $\angle AQR$ is 1, or $m\angle AQR=1$.

To find the degree-measure of $\angle AQB$ by using a protractor:

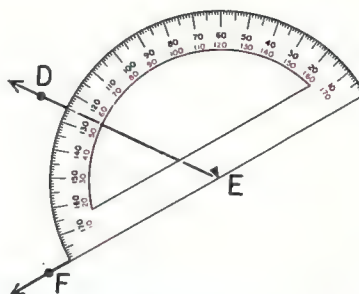
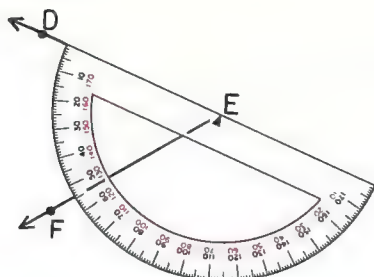
- Place the center of the circle at the vertex (point Q) of the angle.
- Align one side of the angle with the diameter of the circle so that the other side of the angle intersects the semicircle.
- Read the degree-measure of the angle where the other side of the angle intersects the semicircle.

$m\angle AQB=35$. What is $m\angle AQC$? What is $m\angle AQD$? Did you use the inner or the outer scale to read these measures?

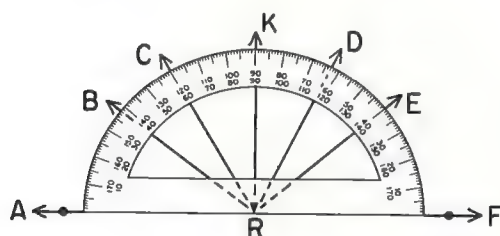
$m\angle EQD=80$. What is $m\angle EQC$? What is $m\angle EQB$? Did you use the inner or the outer scale to read these measures?

Rays QA and QE form a straight line. If you think of \vec{EA} as $\angle EQA$, what is $m\angle EQA$?

A protractor may be placed in either way shown below to find the degree-measure of $\angle DEF$. Notice which scale is used in each case.



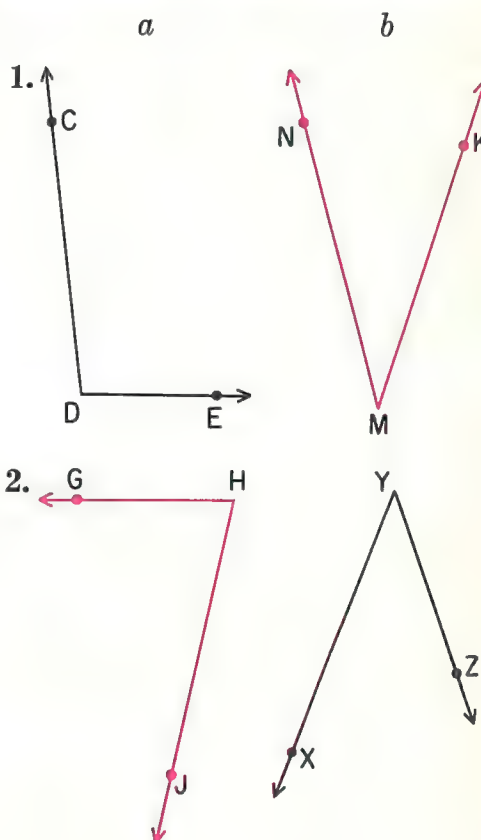
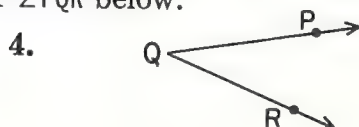
Oral Use the following drawing to state the degree-measure of each angle named below.



Written Use a protractor to find the degree-measure of each angle below.

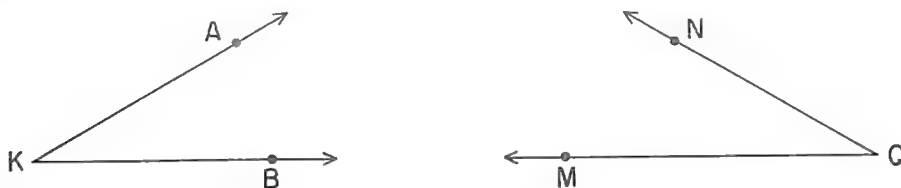
- | | <i>a</i> | <i>b</i> | <i>c</i> |
|----|--------------|--------------|--------------|
| 1. | $\angle FRE$ | $\angle FRD$ | $\angle FRC$ |
| 2. | $\angle FRB$ | $\angle ARB$ | $\angle ARC$ |
| 3. | $\angle ARK$ | $\angle ARD$ | $\angle ARE$ |

Explain how you could place a protractor to find the degree-measure of $\angle PQR$ below.



Congruent Angles

Use a protractor to find $m\angle AKB$ and $m\angle MQN$ below.

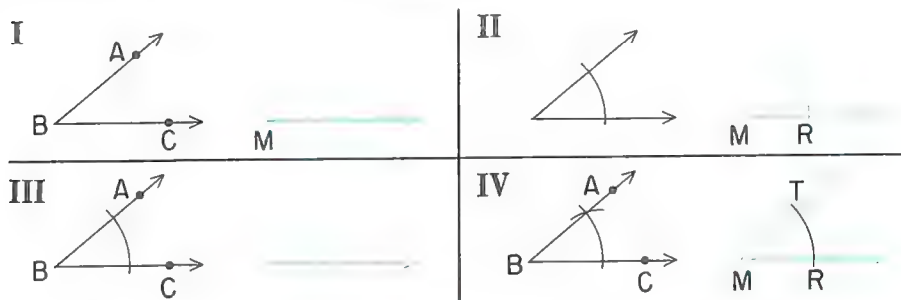


Since $m\angle AKB = m\angle MQN$, we say that $\angle AKB \cong \angle MQN$. We read $\angle AKB \cong \angle MQN$ as $\angle AKB$ is *congruent* to $\angle MQN$. Could you also say that $m\angle MQN = m\angle AKB$? That $\angle MQN \cong \angle AKB$?

For any two angles: If $m\angle ABC = m\angle DEF$, then $\angle ABC \cong \angle DEF$.

If $\angle ABC \cong \angle DEF$, then $m\angle ABC = m\angle DEF$.

By using *only* a compass and a straightedge, you can construct an angle that is congruent to $\angle ABC$ as shown below.



- I. Draw any ray and label its endpoint, like M above. Use point B as the center and any convenient radius to draw an arc that intersects both sides of $\angle ABC$.
- II. With the same radius, use point M as the center and draw an arc that intersects the ray. Label this point of intersection R. This arc must be approximately the same length as the arc drawn in I.
- III. Adjust the compass to the distance between the points where the original arc intersects the sides of $\angle ABC$.
- IV. With the same compass setting, use point R as the center and draw an arc that intersects the ray. Label this point of intersection T. This arc must be approximately the same length as the arc drawn in I.

IV. With this compass opening as the radius, use point R as the center and draw an arc that intersects the previously drawn arc. Label this point of intersection T. Draw ray MT.

By construction, $\angle ABC \cong \angle RMT$. Notice that a protractor is *not* used in this geometric construction. However, you can use a protractor to check the accuracy of this construction.

Oral The measurement of each angle is shown below.

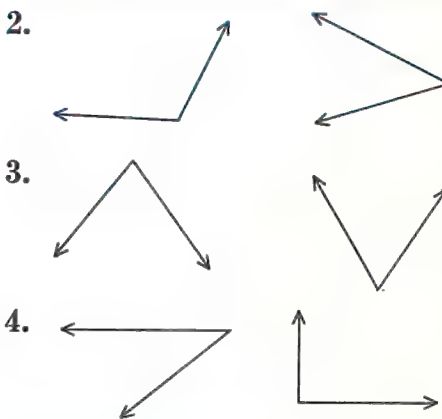
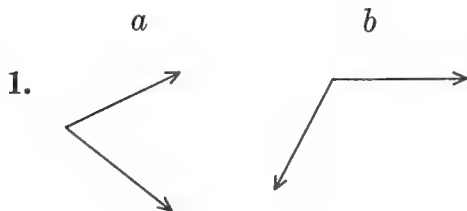


1. What unit of measure is used to find the measurement of each angle above?

2. Which is correct: $\angle DBE = \angle FCG$ or $m\angle DBE = m\angle FCG$? Why?

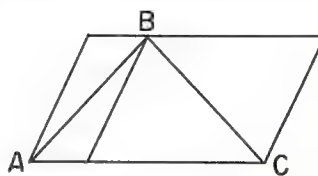
3. What does it mean to say that $\angle DBE \cong \angle FCG$?

Written For each of the following angles, draw an angle that has about the same measure and is in the same position. Then construct an angle that is congruent to each angle you drew.

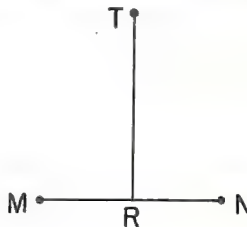


Can you do this? Answer these questions. Then check your answer by measuring.

1. Which is longer: \overline{AB} or \overline{BC} ?



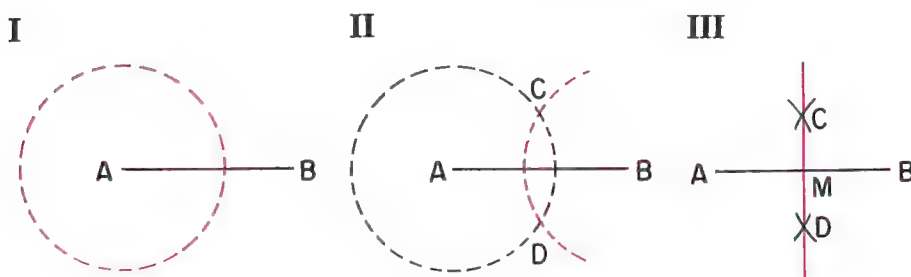
2. Which is longer: \overline{RT} or \overline{MN} ?



Bisecting a Line Segment

To bisect something means to separate it into two parts of the same size. Hence, to bisect \overline{AB} means to locate some point M on \overline{AB} such that $\overline{AM} \cong \overline{MB}$. We then say that point M bisects \overline{AB} or that point M is the **midpoint** of \overline{AB} .

You can bisect \overline{AB} by construction as shown below.



- I. Open a compass to more than half the length of \overline{AB} . Use A as the center and draw an arc as shown.
- II. Using the same compass opening, use B as the center and draw an arc that intersects the previously drawn arc in two points, like C and D above.
- III. Draw \overline{CD} . Label the intersection of \overline{AB} and \overline{CD} as M . By construction, $\overline{AM} \cong \overline{MB}$ so point M bisects \overline{AB} .

Oral Answer questions 1–3.

1. In **I** and **II** above, could the radius be the length of \overline{AB} ? Could it be longer than \overline{AB} ?

2. What would happen if the radius in **I** and **II** were the length of \overline{AM} ? Shorter than \overline{AM} ?

3. Is it necessary to draw the complete arc shown in **I**?

Written Draw line segments having the lengths given below. Bisect each segment by construction.

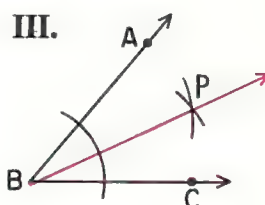
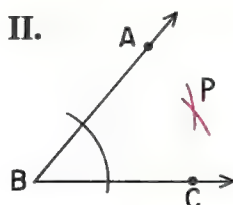
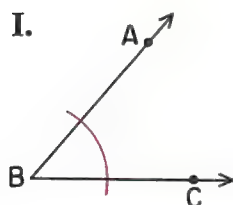
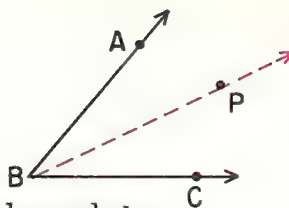
	a	b	c
1.	4 in.	3 in.	5 in.
2.	2 in.	$1\frac{1}{2}$ in.	$3\frac{1}{2}$ in.

Tell how How can $\overline{PR} \cong \overline{RQ}$ be true if R is not the midpoint of \overline{PQ} ?

Bisecting an Angle

To bisect $\angle ABC$ means to locate some ray BP , with point P in the interior of $\angle ABC$, so that $\angle ABP \cong \angle PBC$.

You can bisect $\angle ABC$ by construction as shown below.

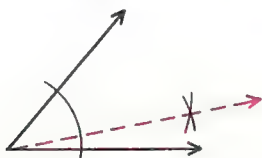


I. Use the vertex of the angle as the center and any convenient radius to draw an arc that intersects both sides of $\angle ABC$.

II. Use the points of intersection in I as centers and the same radius to draw two intersecting arcs in the interior of $\angle ABC$. Label this point of intersection P .

III. Draw \overrightarrow{BP} and you have $\angle ABP \cong \angle PBC$.

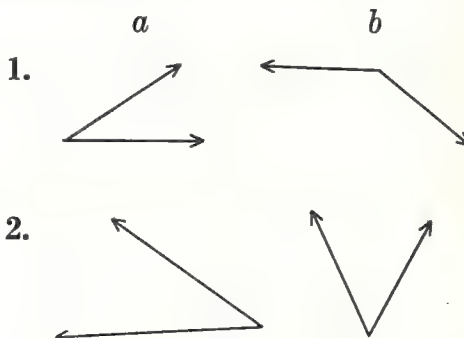
Oral Suppose you use a different radius to draw each of the intersecting arcs in II above.



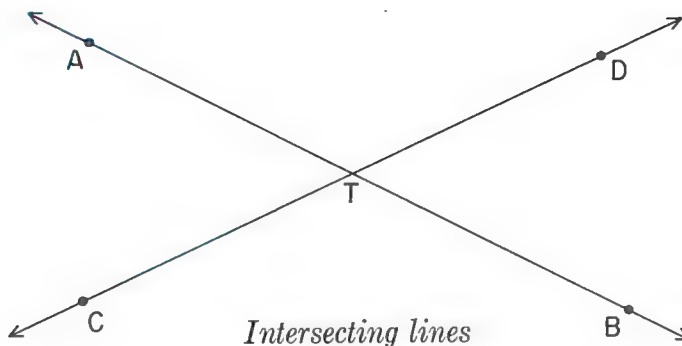
1. Must you use the same radius to draw both of the intersecting arcs when bisecting an angle? Why?

2. Does the radius for drawing the intersecting arcs have to be the same as that for the arc in I?

Written Draw angles having approximately the same measure and in the same position as those below. Then bisect each angle by construction.



Vertical Angles



If two lines are in the same plane, they may or may not intersect. If they do intersect, how many points do they have in common?

Two lines in a plane may intersect in only one point.

In the drawing above, rays TA and TB form a straight line, and so do rays TC and TD. If you think of \overline{AB} as $\angle ATB$, what is the sum of $m\angle ATD$ and $m\angle DTB$? What is the sum of $m\angle ATD$ and $m\angle CTA$?

Recall that a symbol like $m\angle ATD$ names a measure. Since measures are numbers, you can add or subtract measures just as you do numbers. Therefore, you treat a symbol like $m\angle ATD$ in the following sentences just as you do a numeral.

$$m\angle ATD + m\angle DTB = 180$$

$$m\angle DTB = 180 - m\angle ATD$$

$$m\angle ATD + m\angle CTA = 180$$

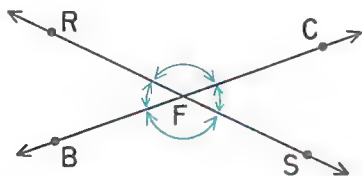
$$m\angle CTA = 180 - m\angle ATD$$

$m\angle DTB$ and $m\angle CTA$ are each equal to $180 - m\angle ATD$, so we can use the transitive property of equality to conclude that $m\angle DTB = m\angle CTA$. Since their measures are the same, we can say that $\angle DTB \cong \angle CTA$.

A pair of angles like $\angle CTA$ and $\angle DTB$ are called **vertical angles**. Name another pair of vertical angles in the figure above.

Vertical angles are congruent.

Oral Use the following figure to answer the questions below.



1. Which angles in the figure are vertical angles?

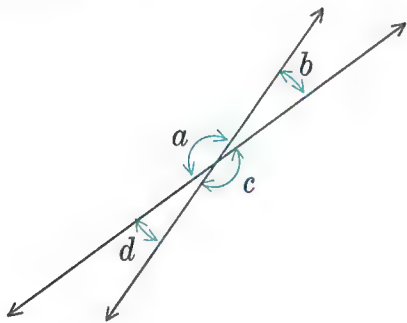
2. If $m\angle RFB = 40$, what is the degree-measure of $\angle CFS$?

3. If $m\angle RFB = 40$, what is the degree-measure of $\angle RFC$?

4. If $m\angle BFS = 150$, what is the degree-measure of $\angle RFC$?

5. If $m\angle BFS = 150$, what is the degree-measure of $\angle CFS$?

Written Two lines intersect in the figure below. Use the figure to complete each sentence so that it becomes true.



1. If $m\angle a = 162$, then $m\angle c = \underline{\hspace{2cm}}$.

2. $\angle a \cong \angle \underline{\hspace{2cm}}$

3. If $m\angle a = 145$, then $m\angle d = \underline{\hspace{2cm}}$.

4. $\angle b \cong \angle \underline{\hspace{2cm}}$

5. $\angle a$ and $\angle \underline{\hspace{2cm}}$ are vertical angles.

6. $\angle d$ and $\angle \underline{\hspace{2cm}}$ are vertical angles.

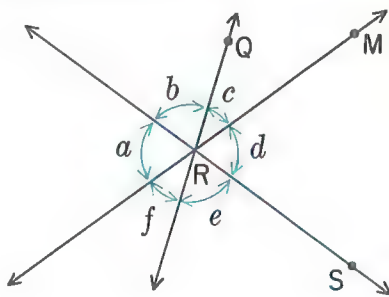
7. $m\angle a + m\angle b = \underline{\hspace{2cm}}$

8. $m\angle a + m\angle b + m\angle c + m\angle d = \underline{\hspace{2cm}}$

9. If $m\angle d = 34$, then $m\angle b = \underline{\hspace{2cm}}$.

10. If $m\angle d = 34$, then $m\angle c = \underline{\hspace{2cm}}$.

Can you do this? Three lines intersect in the figure below. Use the figure to complete each sentence so that it becomes true.



1. If $m\angle a = 40$ and $m\angle c = 50$, then $m\angle b = \underline{\hspace{2cm}}$.

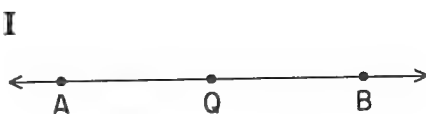
2. If $m\angle f = 45$ and $m\angle d = 60$, then $m\angle b = \underline{\hspace{2cm}}$.

3. If $m\angle a + m\angle f = 112$, then $m\angle c + m\angle d = \underline{\hspace{2cm}}$.

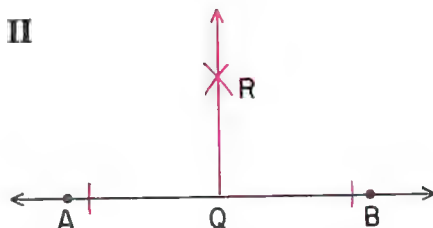
4. If $m\angle e = 70$ and \overline{RM} bisects $\angle QRS$, then $m\angle c = \underline{\hspace{2cm}}$.

Perpendicular Lines

Think of \overleftrightarrow{AB} in I as $\angle AQB$. What is $m\angle AQB$?

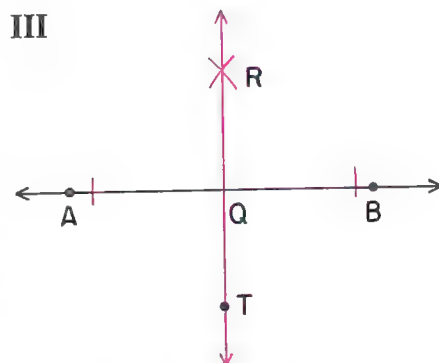


Then you could bisect $\angle AQB$ as shown in II. What is the degree-measure of $\angle AQR$? Of $\angle BQR$? Are these two angles congruent? Why?



An angle like $\angle AQR$ or $\angle BQR$ is called a **right angle**. What is the degree-measure of a right angle? Do you think that all right angles are congruent? Why?

Suppose that after point R is located, you draw *line* QR instead of *ray* QR. This is shown in III. Are any vertical angles formed? Which ones? What is the degree-measure of $\angle AQT$? Of $\angle BQT$? Are all four angles congruent? Why? In such a case we say that \overleftrightarrow{AB} and \overleftrightarrow{RT} are **perpendicular lines**, or that \overleftrightarrow{AB} is *perpendicular to* \overleftrightarrow{RT} , which can be abbreviated as $\overleftrightarrow{AB} \perp \overleftrightarrow{RT}$ or as $\overleftrightarrow{RT} \perp \overleftrightarrow{AB}$.



If two lines intersect so that four right angles are formed, the lines are *perpendicular*.

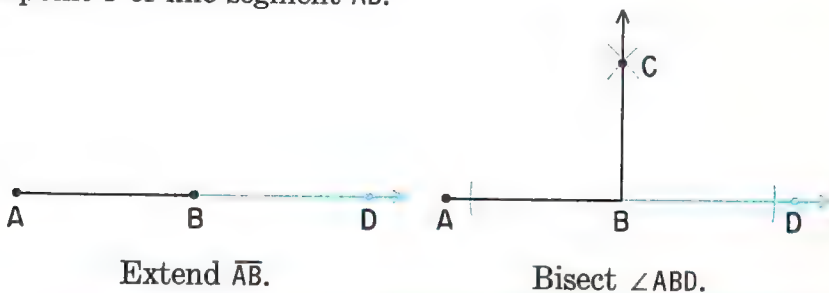
The relationship between perpendicular lines and right angles can also be stated as follows.

If two lines are perpendicular, they intersect so that right angles are formed.

You can consider two line segments to be perpendicular if the lines of which they are parts are perpendicular. In III above, you can say $\overline{AB} \perp \overline{RT}$ because $\overleftrightarrow{AB} \perp \overleftrightarrow{RT}$.

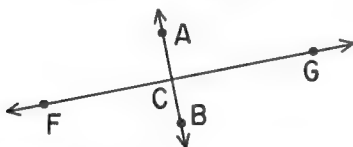
It is sometimes necessary to construct a line perpendicular to a given line at a specified point on the line. As in **III** on the preceding page, simply think of the line as an angle of 180° and bisect that angle.

The figures below show how to construct a right angle at endpoint B of line segment AB.



Oral Assume that you know what is *Given* in each figure below. Then answer questions 1–4.

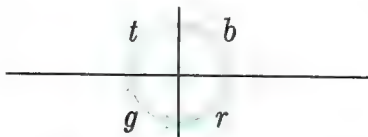
Given: $\overline{AB} \perp \overline{FG}$



1. What is the degree-measure of $\angle FCA$? Of $\angle FCB$? Of $\angle BCG$?

2. What kind of angle is $\angle FCA$?

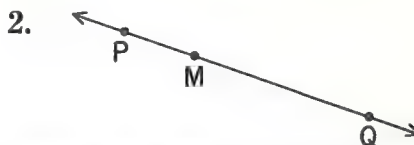
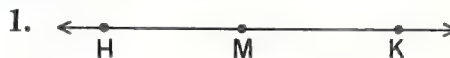
Given: $\angle b$ is a right angle.



3. What kind of angle is $\angle r$?

4. Are the two line segments perpendicular? Why or why not?

Written Draw and label lines as shown below. Then construct a line perpendicular to each line at point M.



Draw and label line segments as shown below. Then construct a right angle at point C on each line segment.



Tell why Suppose that $\overline{AB} \perp \overline{CD}$ and $\overline{CD} \perp \overline{EF}$. Draw a figure to show that \overline{AB} is not perpendicular to \overline{EF} .

Classification of Angles

The figure at the right is so constructed that $\angle AVC$ is a right angle. What is its degree-measure?

$\angle AVB$ is called an **acute angle**. How does $m\angle AVB$ compare with $m\angle AVC$?

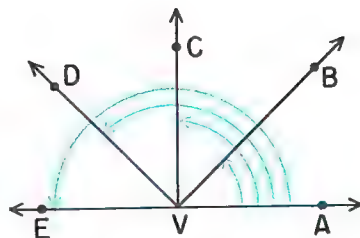
$\angle AVD$ is called an **obtuse angle**. How does $m\angle AVD$ compare with $m\angle AVC$? Is $m\angle AVD$ less than 180° ?

$\angle AVE$ is sometimes considered to be a **straight angle**. What is the degree-measure of $\angle AVE$?

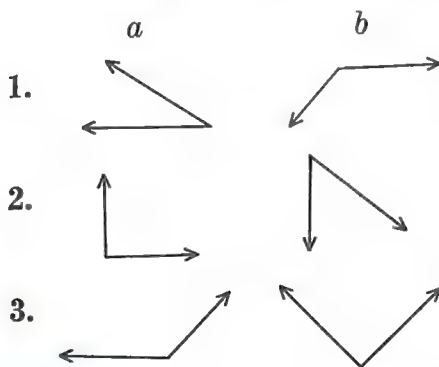
The degree-measure of an *acute angle* is between 0 and 90 .

The degree-measure of an *obtuse angle* is between 90 and 180 .

The degree-measure of a *straight angle* is 180 .



Oral Without measuring them, tell whether each angle below is an acute, a right, or an obtuse angle.



Written For each of the following statements, write T if it is true and write F if it is false. For each answer F, draw a figure to verify your answer.

1. If one of two vertical angles is acute, the other is also acute.

2. The sum of the measures of two acute angles is greater than 90 .

3. The sum of the measures of an acute angle and an obtuse angle is 180 .

4. If either side of an acute angle is extended into the exterior of the angle, an obtuse angle is formed by this extension and the other side of the angle.

5. If a right angle is bisected, each new angle is an acute angle.

6. If $m\angle ABC = 70$ and $m\angle CBD = 30$, then $m\angle ABD = 100$.

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Through two points there is one and only one line. (208)

2. To make a geometric construction, only a straightedge and a compass are used. (212)

3. Congruent line segments have the same length. (212)

4. Congruent angles have the same degree-measure. (218)

5. Two lines in a plane may intersect in only one point. (222)

6. Vertical angles are congruent. (222)

Words to Know

1. Line (207)

2. Line segment, endpoints, ray, plane (208)

3. Circle, radius, chord, diameter (210)

4. Arc, semicircle (211)

5. Congruent line segments (212)

6. Angle, vertex (214)

7. Degree, degree-measure (216)

8. Congruent angles (218)

9. Vertical angles (222)

10. Perpendicular lines, right angle (224)

11. Acute angle, obtuse angle, straight angle (226)

Questions to Discuss

1. How many endpoints has a line? A line segment? A ray? (207, 208)

2. How would you place a protractor to find the degree measure of an angle? (216)

3. If two lines are perpendicular, what can you conclude about the angles that are formed? (224)

4. If two lines intersect to form right angles, what can you conclude about the two lines? (224)

Written Practice

1. Draw \overline{AB} about 2 inches long. Construct \overline{CD} congruent to \overline{AB} . Then bisect \overline{AB} . (212, 220)

2. Draw acute angle $\angle FGH$. Construct $\angle PQR$ congruent to $\angle FGH$. Then bisect $\angle FGH$. (218, 221)





3. Draw \overline{MN} . Construct \overline{DN} perpendicular to \overline{MN} at point N . (224)

Self-Evaluation


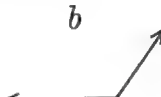

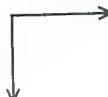
Part 1 Draw and label a picture of each geometric figure named below.

- | a | b |
|--|--------------------|
| 1. \overline{RM} | point K |
| 2. \overleftrightarrow{AB} | circle T |
| 3. \overline{BA} | chord RS |
| 4. \overline{DE} | radius TA |
| 5. $\angle MNL$ | diameter AC |
| 6. $\angle LMN$ | arc RC |
| 7. $\overline{CD} \perp \overline{DE}$ | right $\angle ABC$ |

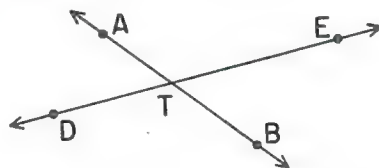
Part 2 Draw line segments having approximately the same length as those below. Construct a line segment congruent to each of them. Then bisect the line segments that you have constructed.

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- 
- 

Draw angles having approximately the same measures as those below. Construct an angle congruent to each of them. Then bisect each angle you have constructed.

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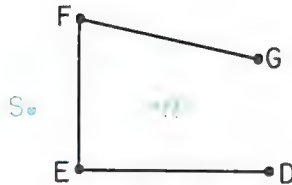
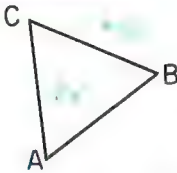
Part 3 Lines AB and DE intersect in the figure below. Complete each sentence so that it becomes true.



- If $m\angle ATE = 100$, then $m\angle DTB = \underline{\hspace{1cm}}$ and $m\angle DTA = \underline{\hspace{1cm}}$.
- If $m\angle DTB = 90$, then $\overline{DE} \perp \underline{\hspace{1cm}}$.
- Angles ATD and BTE are called $\underline{\hspace{1cm}}$ angles.
- If $\angle BTE$ is an acute angle, then $\angle ATD$ is an $\underline{\hspace{1cm}}$ angle and $\angle DTB$ is an $\underline{\hspace{1cm}}$ angle.
- $m\angle DTB + m\angle BTE = \underline{\hspace{1cm}}$

Chapter 10 POLYGONS

Open and Closed Figures



Could you draw figure ABC by beginning and ending at the same point without lifting the pencil? Point P is in the interior of figure ABC and point Q is in its exterior. Can you draw a continuous path from P to Q that will *not* intersect at least one of the sides of figure ABC?

Now consider the figure formed by line segments DE, EF, and FG. Can you draw a continuous path from R to S that will not intersect at least one of the line segments?

A figure like ABC is called a **closed figure**. A figure like DEFG is called an **open figure**. How would you define each type of figure?

Oral Identify each figure below as an *open figure* or a *closed figure*.

Written Draw the following.

- | | a | b | c |
|----|---|---|---|
| 1. | | | |
| 2. | | | |
| 3. | | | |

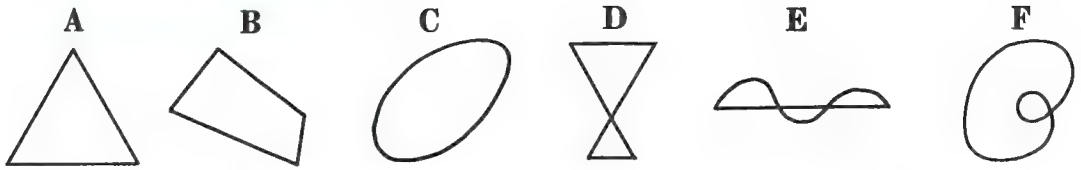
1. 5 open figures formed by line segments

2. 5 closed figures formed by line segments

3. 5 open figures that contain no line segments

4. 5 closed figures that contain no line segments

Simple Closed Figures

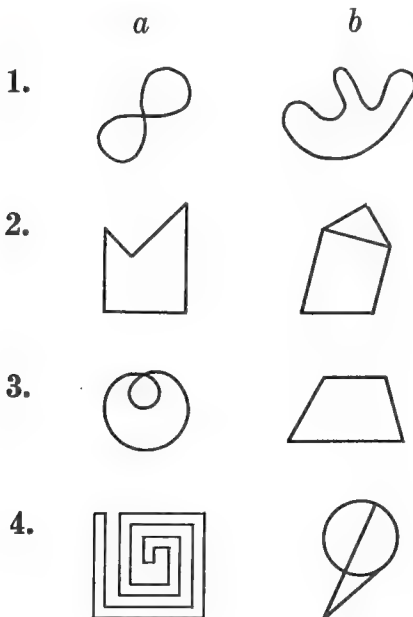


Why is each figure above a closed figure? Which of these figures can you draw without having the figure intersect itself? Which figures have only 1 interior region?

Figures like **A**, **B**, and **C** are closed figures that separate the plane into 3 sets of points—the points on the figure, in the interior, and in the exterior. They are called **simple closed figures**.

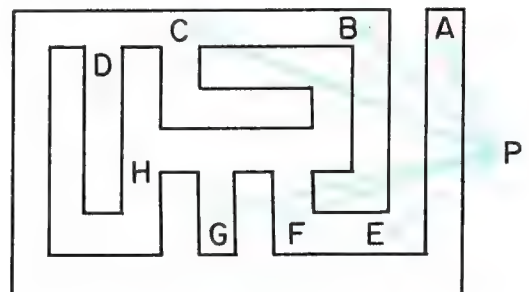
Why are figures **D**, **E**, and **F** not simple closed figures?

Oral Tell whether or not each figure below is a simple closed figure.

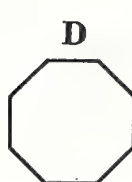
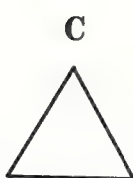
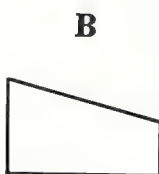
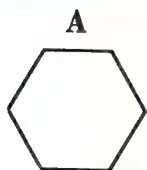


Written Draw 5 simple closed figures. Then draw 5 closed figures that are not simple.

Can you do this? **A**, **B**, and **C** are in the interior of the closed figure below and **E**, **F**, **G**, and **P** are in its exterior. How many times does each blue line segment intersect the figure? What pattern do you discover? Are **D** and **H** in the interior or in the exterior?



Polygons



Why is each figure above a simple closed figure? Is each figure formed by line segments?

A simple closed figure formed by line segments is called a **polygon**. Each line segment is called a **side** of the polygon.

Polygons can be classified according to the number of sides they contain. Some common polygons are classified below.

<i>Polygon</i>	<i>Number of sides</i>	<i>Polygon</i>	<i>Number of sides</i>
Triangle	3	Hexagon	6
Quadrilateral	4	Heptagon	7
Pentagon	5	Octagon	8

Name each polygon shown at the top of this page.

Oral Answer these questions.

1. How many sides are there in figure A above?
2. How many angles does figure A contain?
3. How does the number of sides in figure A compare with the number of angles it contains?
4. Is the number of sides of a polygon always the same as the number of angles it contains?

Written Draw two different polygons of each kind named below.

a

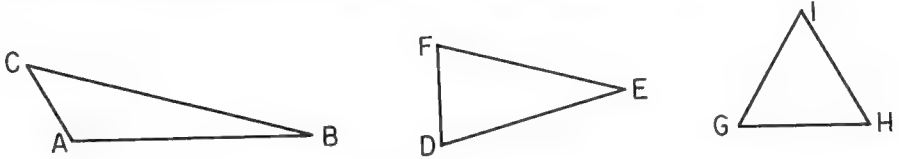
b

- | | |
|-------------|---------------|
| 1. triangle | quadrilateral |
| 2. hexagon | octagon |
| 3. pentagon | heptagon |

Tell why Can you draw a simple closed figure in a plane by drawing only 2 line segments? Why must a polygon have at least 3 sides?

Triangles

The first figure below can be named *triangle* ABC. Triangle ABC can be denoted by $\triangle ABC$. Triangles can be classified according to the number of congruent sides they have.

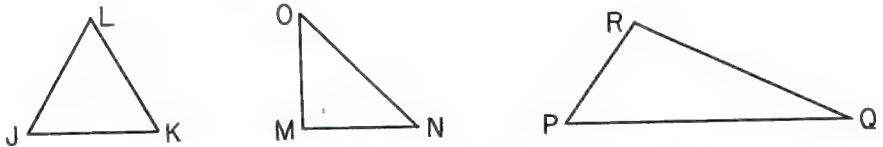


None of the sides of $\triangle ABC$ are congruent. A triangle like $\triangle ABC$ that has no congruent sides is called a **scalene** (skā lēn') **triangle**.

In $\triangle DEF$, $\overline{FE} \cong \overline{DE}$. A triangle like $\triangle DEF$ that has two congruent sides is called an **isosceles** (ī sōs' ē lēz) **triangle**.

In $\triangle GHI$, $\overline{GH} \cong \overline{HI} \cong \overline{IG}$. A triangle like $\triangle GHI$ that has three congruent sides is called an **equilateral triangle**. In the term equilateral, *equi* means equal in length or congruent and *lateral* means with regard to sides.

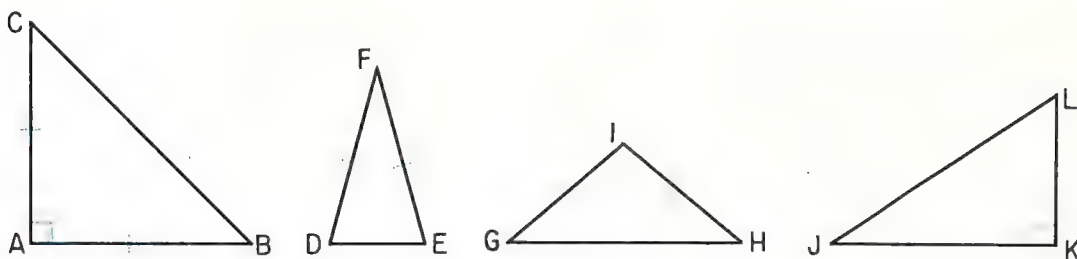
Triangles can also be classified according to the kinds of angles they contain.



What kind of angles are all of the angles in $\triangle JKL$? Triangle JKL contains 3 acute angles and is called an **acute triangle**. If all 3 angles happen to be congruent, the triangle is called an **equiangular triangle**.

The \angle in $\triangle MNO$ indicates that $\angle OMN$ is a right angle. What kind of angles are the other two angles? Triangle MNO contains 1 right angle and is called a **right triangle**.

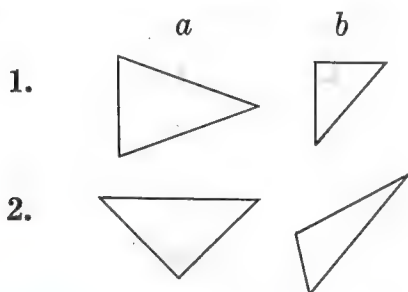
What kind of angles does $\triangle PQR$ contain? Triangle PQR contains 1 obtuse angle and is called an **obtuse triangle**.



The small blue marks on the sides of $\triangle ABC$ indicate that $\overline{AB} \cong \overline{AC}$. According to its sides, what kind of triangle is $\triangle ABC$? According to its angles, what kind of triangle is it? For a complete classification, you can call $\triangle ABC$ a **right isosceles triangle**.

What is the complete classification for each of the other triangles shown above?

Oral Classify each triangle below according to its sides, according to its angles, and finally make a complete classification.



Written Draw a triangle for each classification below.

- | | |
|-----------|------------------|
| <i>a</i> | <i>b</i> |
| 1. acute | right scalene |
| 2. obtuse | obtuse isosceles |
| 3. right | equilateral |

Do the following.

4. Construct a triangle that has two congruent sides. Measure the angles opposite the congruent sides. What seems to be true about the angles opposite the congruent sides of a triangle?

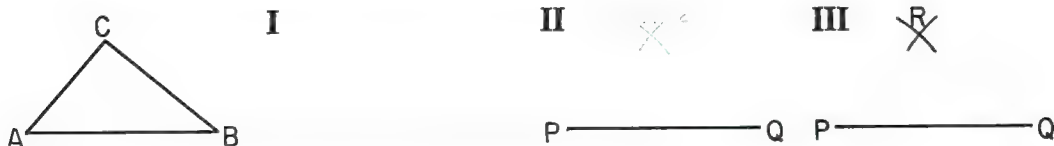
5. Construct an equilateral triangle. Measure each of the angles. What seems to be true about the angles of an equilateral triangle?

6. Construct any triangle. Measure its angles. Add the measures of all 3 angles. What seems to be true about the sum of the measures of the angles of a triangle?

Tell why All equilateral triangles are isosceles, but not all isosceles triangles are equilateral.

Duplicating a Triangle

To duplicate a triangle means to construct a triangle having its sides and its angles congruent to those of a given triangle. There are three basic ways to duplicate a triangle. One way to duplicate $\triangle ABC$ is explained below.



- I. Construct \overline{PQ} so that $\overline{PQ} \cong \overline{AB}$. We say that point P corresponds to point A, that point Q corresponds to point B, and that \overline{PQ} corresponds to \overline{AB} .
- II. With P as the center and \overline{AC} as a radius, draw arc 1. With Q as the center and \overline{BC} as a radius, draw arc 2. The point of intersection of these arcs is labeled R. To which point in $\triangle ABC$ does R correspond?
- III. Draw \overline{PR} and \overline{QR} . To which side in $\triangle ABC$ does \overline{PR} correspond? \overline{QR} correspond?

Lay a piece of thin paper over $\triangle PQR$ and trace it. Now lay your tracing over $\triangle ABC$. Can your tracing be placed so that it fits exactly on $\triangle ABC$? Which angle of $\triangle PQR$ appears to be congruent to $\angle ABC$? To $\angle CAB$? To $\angle BCA$?

We say that $\triangle ABC$ is congruent to $\triangle PQR$ or that $\triangle PQR$ is congruent to $\triangle ABC$. This can be denoted by either

$$\triangle ABC \cong \triangle PQR \quad \text{or} \quad \triangle PQR \cong \triangle ABC.$$

In this notation, corresponding points are given in the same order in naming each triangle.

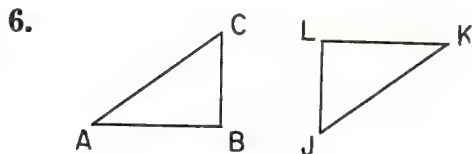
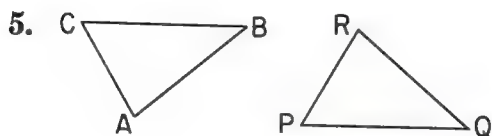
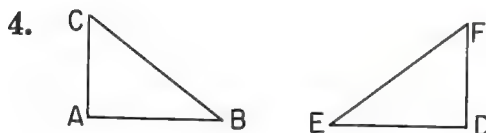
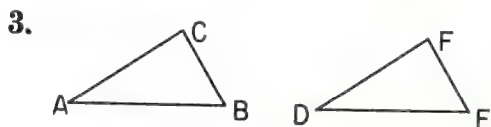
Two triangles are congruent if their corresponding sides are congruent.

Oral Refer to the construction on page 234 to answer the following questions.

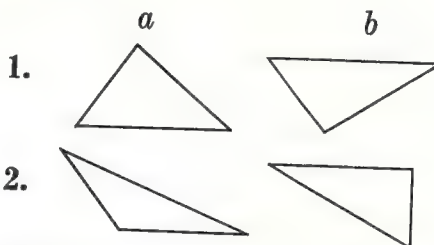
1. Each side of $\triangle PQR$ is so constructed that it is congruent to its corresponding side of $\triangle ABC$. Does it appear that the corresponding angles are also congruent?

2. If the 3 sides of one triangle are congruent to the 3 sides of another triangle, do you think the triangles are congruent?

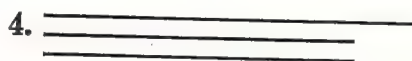
In each exercise below, the two triangles are congruent. Tell which angles and also which sides are corresponding parts.



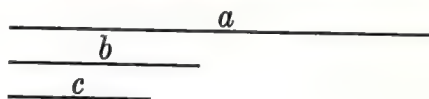
Written Draw triangles like those below. For each triangle you draw, construct a triangle that is congruent to it.



Construct a triangle having sides that are congruent to the line segments given below.



Can you do this? Attempt to construct a triangle having sides congruent to the line segments below.



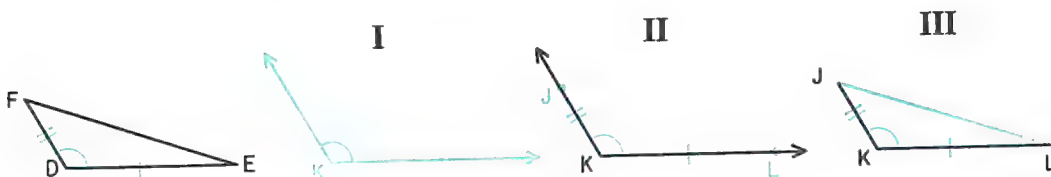
What do you discover?

Is the sum of the measures of line segments b and c less than, equal to, or greater than the measure of line segment a ?

What do you conclude about the sum of the measures of two sides of a triangle with regard to the measure of the third side?

Two Sides and the Included Angle

In $\triangle DEF$ below, which angle is formed by sides DE and DF ? We say that $\angle FDE$ is included by these sides. By using only two sides and the included angle, you can construct a triangle that is congruent to a given triangle.



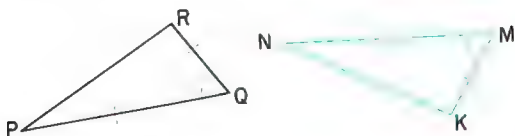
- I. Construct $\angle K$ so that $\angle K \cong \angle FDE$.
- II. Construct \overline{KL} so that $\overline{KL} \cong \overline{DE}$. This is indicated by the single small marks on \overline{DE} and \overline{KL} . Construct $\overline{KJ} \cong \overline{DF}$. How is this indicated above?
- III. Draw \overline{JL} .

Make a tracing of $\triangle JKL$ and see if it will fit exactly on $\triangle FDE$. Do you think these two triangles are congruent?

$$\triangle JKL \cong \triangle FDE$$

Two triangles are congruent if two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle.

Oral Explain the construction below. State a sentence like $\triangle ABC \cong \triangle DEF$ for the pair of triangles. (Remember to give the corresponding parts in the same order.)



Written Do the following.

1. Draw a triangle on your paper. Use a construction like that above to construct a triangle congruent to the one you drew.

2. Construct a right triangle so that sides of 2" and 3" include the right angle.

Two Angles and the Included Side

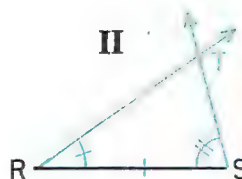
In $\triangle ABC$ below, we say that angles CAB and CBA *include* side AB . Which angles *include* side BC ? Side AC ? By using only two angles and the included side, you can construct a triangle that is congruent to a given triangle.



I



II



I. Construct \overline{RS} so that $\overline{RS} \cong \overline{AB}$.

II. Construct $\angle R$ so that $\angle R \cong \angle A$. This is indicated by the single marks on the arcs within the angles. Construct $\angle S \cong \angle B$. How is this indicated? The point of intersection of the two rays is labeled T .

Make a tracing of $\triangle RST$ and see if it will fit exactly on $\triangle ABC$. Do you think the triangles are congruent?

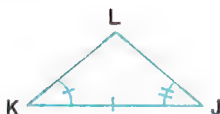
$$\triangle RST \cong \triangle ABC$$

Two triangles are congruent if two angles and the included side of one triangle are congruent to two angles and the included side of the other triangle.

Oral Answer questions 1–2.

1. In II above, why should the two rays be drawn on the same side of RS ?

2. Explain how $\triangle JKL$ is constructed congruent to $\triangle DEF$ below.

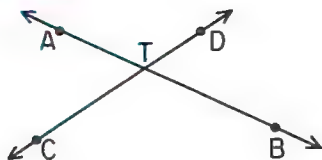


Written Do the following.

1. Draw a triangle on your paper. Use a construction like that above to construct a triangle congruent to the one you drew.

2. Construct a triangle having two angles 45° each and the side between them 2" long. Do not use a protractor. (*Hint:* Bisect a right angle.)

Parallel Lines



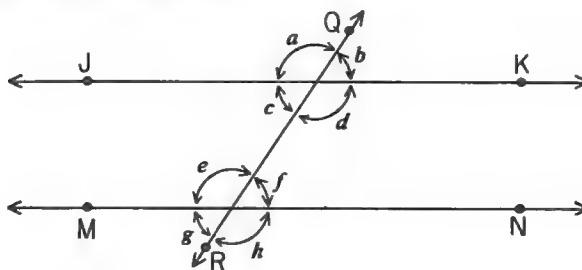
\overleftrightarrow{AB} and \overleftrightarrow{CD} intersect in point T . Which angles are congruent? Why?

Suppose that \overleftrightarrow{EF} and \overleftrightarrow{GH} are in the same plane and never intersect. Since we cannot draw a complete line, let us agree that the drawing above denotes that the two lines never intersect. Such lines are called **parallel lines**.

Two lines in the same plane that do not intersect are called *parallel lines*. Line EF is parallel to line GH can be denoted by $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$.

Two line segments are said to be parallel if the lines of which they are parts are parallel. $\overline{EF} \parallel \overline{GH}$ because $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$.

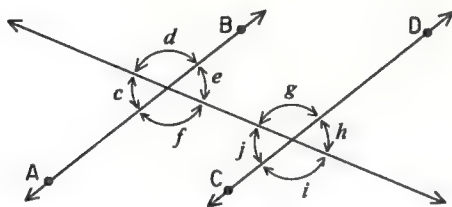
\overleftrightarrow{QR} is called a **transversal** of parallel lines JK and MN because it intersects both of the lines.



What kind of angles are $\angle a$ and $\angle d$? What do you know about their measures? Find three more pairs of vertical angles in the figure.

The pairs of angles named by a and e , by b and f , by c and g , and by d and h are called **corresponding angles**. Corresponding angles on parallel lines have the same measure. Therefore, $m\angle a = m\angle e$, $m\angle b = m\angle f$, and so on.

Oral $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ in the figure below. Give a reason for each statement below the figure.

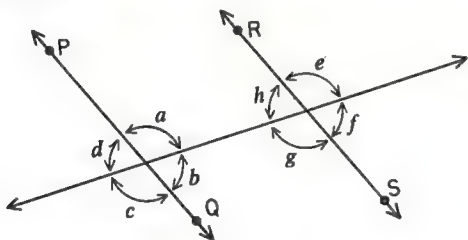


a

b

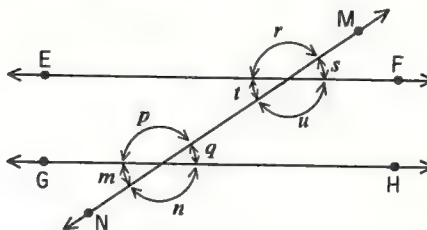
- $\angle c \cong \angle e$ $\angle d \cong \angle f$
- $m\angle c = m\angle e$ $m\angle d = m\angle f$
- $m\angle c = m\angle j$ $m\angle d = m\angle g$
- $m\angle e = m\angle h$ $m\angle f = m\angle i$

Use the figure below to complete sentences 5–9 correctly. Assume that $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$.



- If $m\angle d = 60$, then $m\angle b = \underline{\hspace{1cm}}$.
- If $m\angle d = 60$, then $m\angle a = \underline{\hspace{1cm}}$.
- If $m\angle a = 110$, then $m\angle e = \underline{\hspace{1cm}}$.
- If $m\angle g = 130$, then $m\angle c = \underline{\hspace{1cm}}$.
- $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$ because $\underline{\hspace{1cm}} \parallel \underline{\hspace{1cm}}$.

Written Use the figure below to complete sentences 1–4 correctly. Assume that $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$.



- $\angle r$ and $\angle \underline{\hspace{1cm}}$ are vertical angles.
- If $m\angle m = 70$, then $m\angle q = \underline{\hspace{1cm}}$.
- If $m\angle u = 100$, then $m\angle n = \underline{\hspace{1cm}}$.
- $m\angle r + m\angle s = \underline{\hspace{1cm}}$.

Use the figure above to supply a reason for each statement below.

- $m\angle s = m\angle q$
- $m\angle r = m\angle u$
- $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$

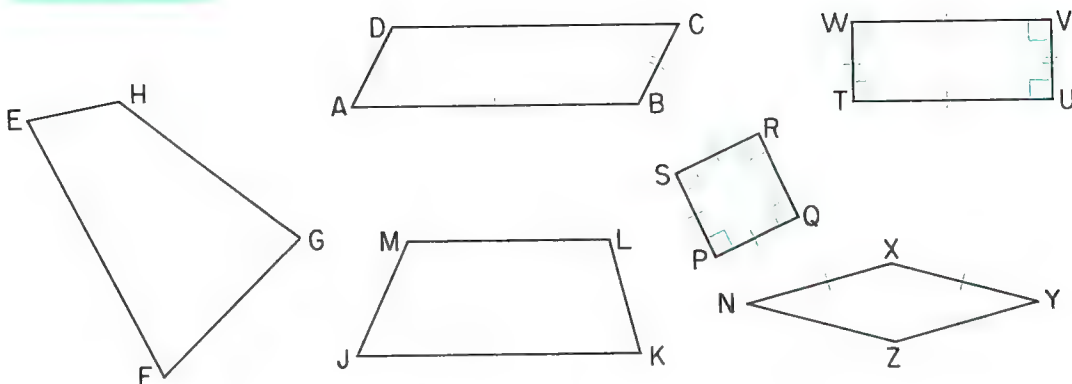
Can you do this? Do you think the following statement is true or false?

If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$.

This property of parallel lines is the same as which property of equality?

Tell why Draw lines CD , MN , and QR in the same plane so that $\overleftrightarrow{CD} \parallel \overleftrightarrow{MN}$ and $\overleftrightarrow{QR} \perp \overleftrightarrow{CD}$. Why is \overleftrightarrow{QR} also perpendicular to \overleftrightarrow{MN} ?

Quadrilaterals



Is each figure above a simple closed figure? Is each figure formed by line segments? Each figure has how many sides? How many angles? Each figure above is called a **quadrilateral**.

A *quadrilateral* is a polygon with 4 sides.

Quadrilateral JKLM has one special feature: $\overline{JM} \parallel \overline{LK}$. A quadrilateral like JKLM with only one pair of opposite sides parallel is called a **trapezoid**.

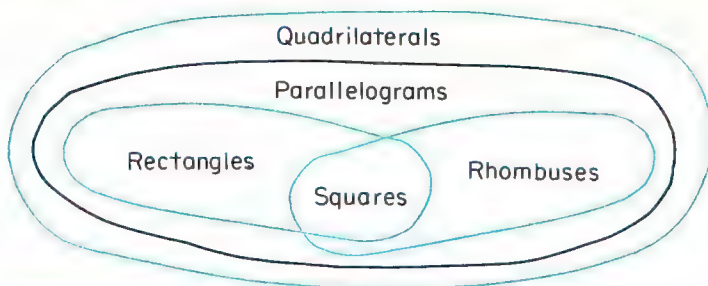
Quadrilateral ABCD has both pairs of opposite sides parallel. Such a quadrilateral is called a **parallelogram**. It also happens that opposite sides of a parallelogram are congruent ($\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{DC}$) and that opposite angles are congruent ($\angle A \cong \angle C$ and $\angle B \cong \angle D$).

Quadrilateral TUVW has all the features of a parallelogram and also has 4 right angles. Such a quadrilateral is called a **rectangle**. Is every rectangle a parallelogram?

Quadrilateral PQRS has all the features of a rectangle and also has 4 congruent sides. Such a quadrilateral is called a **square**. Is every square a rectangle?

Quadrilateral NZYX has all the features of a parallelogram and also has 4 congruent sides. Such a quadrilateral is called a **rhombus**. Is every rhombus a parallelogram?

Some relationships between the types of quadrilaterals are illustrated by the drawing below.



Oral State a reason for the truth of each statement below.

1. Every parallelogram is a quadrilateral, but not every quadrilateral is a parallelogram.

2. Every rectangle is a parallelogram, but not every parallelogram is a rectangle.

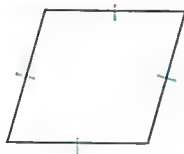
3. Every square is a rectangle, but not every rectangle is a square.

4. If figure ABCD is a rectangle, then it is a parallelogram.

5. If figure ABCD is a square, then it is a rectangle.

6. If figure ABCD is a rhombus, it may or may not be a square.

7. Figure QRST below is a rhombus but not a square.



Written For each quadrilateral below, write sentences to tell which sides are parallel, which sides are congruent, and which angles are congruent. For example, for 1a you will write:

$$\overline{AD} \parallel \overline{BC}$$

$$\overline{AD} \cong \overline{BC}$$

$$\overline{AB} \parallel \overline{DC}$$

$$\overline{AB} \cong \overline{DC}$$

$$\angle A \cong \angle B \cong \angle C \cong \angle D$$

a

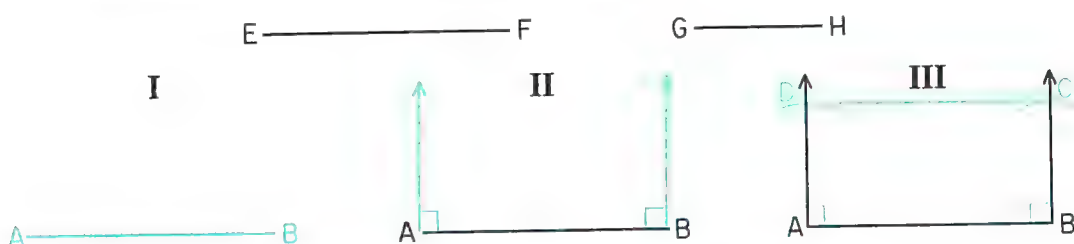
b

- 1.
- 2.
- 3.

Constructing a Rectangle

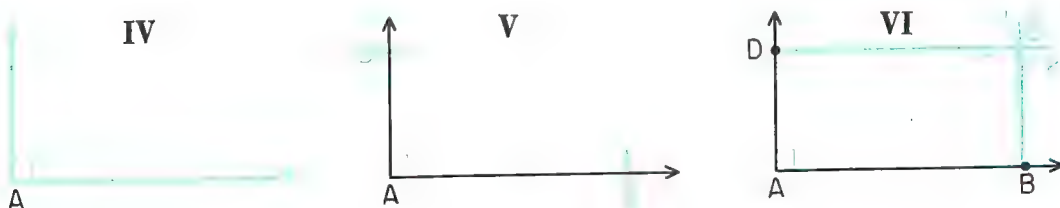
There are two easy ways to construct a rectangle when you are given the lengths of the sides. What do you know about the angles of a rectangle? About the sides of a rectangle?

Let \overline{EF} and \overline{GH} below be two of the given sides. Construct a rectangle whose sides have these lengths.



- I. Construct a line segment, say \overline{AB} , so that $\overline{AB} \cong \overline{EF}$.
- II. Construct right angles at the endpoints of \overline{AB} so that both rays point in the same direction.
- III. Construct \overline{AD} and \overline{BC} so that $\overline{AD} \cong \overline{GH}$ and $\overline{BC} \cong \overline{GH}$. Draw \overline{DC} . Rectangle $ABCD$ has sides as specified above.

Another way to construct such a rectangle is explained below.



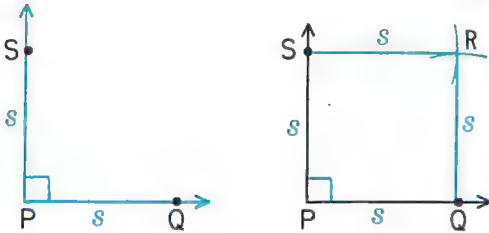
- IV. Construct a right angle. Call it $\angle A$.
- V. Construct \overline{AB} so that $\overline{AB} \cong \overline{EF}$. Construct $\overline{AD} \cong \overline{GH}$.
- VI. With \overline{EF} as a radius and D as the center, draw arc 1. With \overline{GH} as a radius and B as the center, draw arc 2. The intersection of these arcs is labeled C . Draw \overline{CD} and \overline{BC} . Rectangle $ABCD$ has sides as specified above.

Oral Answer these questions.

1. Which method of constructing a rectangle on page 242 do you prefer? Why?

2. Is a square a rectangle? Can you use either method on page 242 to construct a square?

3. Explain how square PQRS is constructed below, given the length of side s .



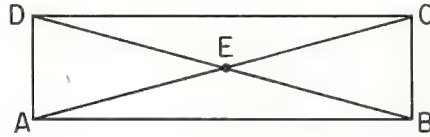
Written Construct rectangles having sides of the given lengths below.

1. _____
2. _____
3. _____

For each line segment below, construct a square having sides of the given length.

4. _____
5. _____
6. _____

Can you do this? Line segments AC and BD are called *diagonals* of rectangle ABCD below. A diagonal is a line segment joining any two non-consecutive vertices of a polygon.



1. Compare the lengths of \overline{AC} and \overline{BD} . What do you discover?

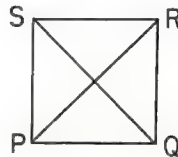
2. Compare the lengths of \overline{AE} and \overline{EC} . What do you discover? Does \overline{BD} bisect \overline{AC} ?

3. Do the diagonals of a rectangle bisect each other?

4. How can you use this fact to show that $\triangle AED \cong \triangle BEC$?

5. If you were given $\triangle AED$, how can you construct rectangle ABCD?

Measure the angles formed by the diagonals in square PQRS below.



6. Are the diagonals of a square perpendicular to each other?

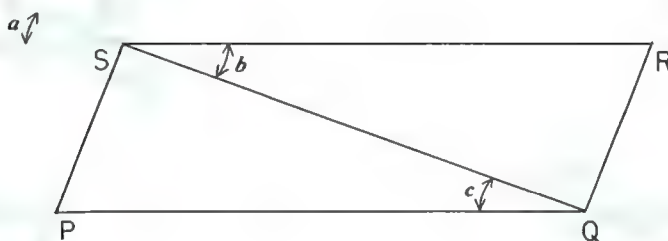
7. If you were given a diagonal of a square, how would you construct the square?

Parallelograms

Figure PQRS below is a parallelogram. Therefore, you know that $\overline{PS} \parallel \overline{QR}$ and that $\overline{PQ} \parallel \overline{SR}$. In an earlier lesson we mentioned that opposite sides of a parallelogram are congruent. Let us try to prove that statement.

Let us draw diagonal SQ in parallelogram PQRS. Notice that opposite sides of the parallelogram are corresponding sides of $\triangle PQS$ and $\triangle RSQ$. If we can show that $\triangle PQS \cong \triangle RSQ$, then we can conclude that opposite sides of a parallelogram are congruent.

As shown below, you can think of the sides as parallel lines and transversals.



$$m\angle c = m\angle a$$

corresponding angles

$$m\angle c = m\angle a$$

$$m\angle b = m\angle d$$

vertical angles

$$m\angle b = m\angle d$$

$$m\angle c = m\angle b$$

transitive property of =

$$m\angle c = m\angle b$$

$$\angle c \cong \angle b$$

Why?

$$\angle c \cong \angle b$$

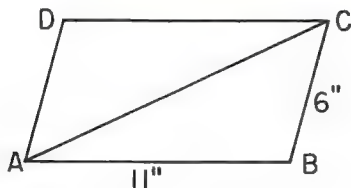
Also, $\overline{QS} \cong \overline{SQ}$. Why?

Do we have two angles and the included side of $\triangle PQS$ congruent to two angles and the included side of $\triangle RSQ$? Are the two triangles congruent?

Since $\triangle PQS \cong \triangle RSQ$, their corresponding parts are congruent. Which side of $\triangle RSQ$ corresponds to side PQ? To side PS? Therefore, $\overline{PQ} \cong \overline{RS}$ and $\overline{PS} \cong \overline{RQ}$.

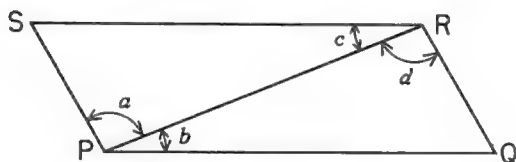
Opposite sides of a parallelogram are congruent.

Oral Figure ABCD below is a parallelogram. Use it to answer questions 1–3.



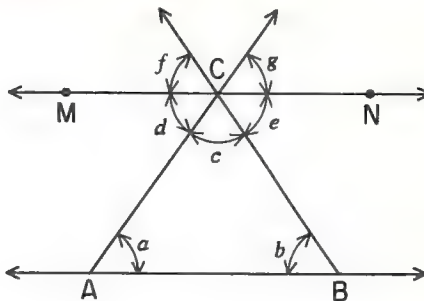
- Which sides of figure ABCD are opposite sides?
- If the inch measure of side AB is 11, what is the inch measure of side DC? Why?
- If the inch measure of side BC is 6, what is the inch measure of side AD? Why?

Written Copy and complete each sentence below. Figure PQRS is a parallelogram.



- If $m\angle S = 42$, then $m\angle Q = \underline{\hspace{1cm}}$.
- If \overline{PQ} is 10" long, then \overline{SR} is $\underline{\hspace{1cm}}$ " long.
- Side PS of $\triangle PRS$ corresponds to side $\underline{\hspace{1cm}}$ of $\triangle RPQ$.
- If $m\angle b = 37$, then $m\angle c = \underline{\hspace{1cm}}$.
- If \overline{SP} is 3 cm. long, then \overline{QR} is $\underline{\hspace{1cm}}$ cm. long.

Can you do this? In the figure below, $\overleftrightarrow{MN} \parallel \overleftrightarrow{AB}$. Rays AC and BC intersect at point C on \overleftrightarrow{MN} .



Think of rays AC and BC as transversals of the parallel lines. Let us try to find the sum of the measures of the angles of $\triangle ABC$. State a reason for each statement.

- $m\angle b = m\angle f$
- $m\angle e = m\angle g$
- Therefore, $m\angle b = m\angle e$.
- $m\angle a = m\angle g$
- $m\angle d = m\angle g$
- Therefore, $m\angle a = m\angle d$.
- $m\angle d + m\angle c + m\angle e = 180$
- $m\angle a + m\angle c + m\angle b = 180$

Notice that angles a , b , and c are the angles contained in $\triangle ABC$.

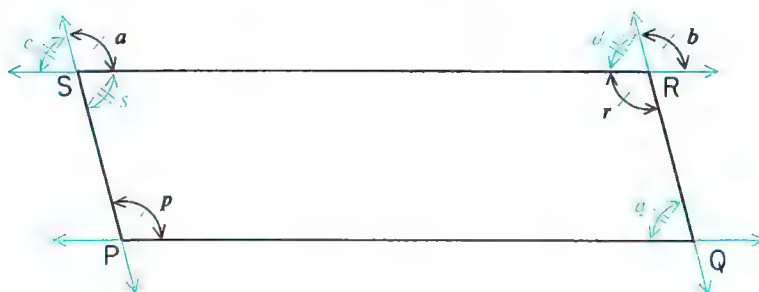
If $m\angle a = 40$ and $m\angle b = 70$, what is $m\angle c$?

If $m\angle c = 85$ and $m\angle b = 65$, what is $m\angle a$?

Parallelograms

In an earlier lesson we mentioned that opposite angles of a parallelogram are congruent. Let us try to prove that statement.

Figure PQRS below is a parallelogram.



$$m\angle r = \angle b$$

vertical angles

$$m\angle s = m\angle c$$

$$m\angle a = m\angle b$$

corresponding angles

$$m\angle d = m\angle c$$

$$m\angle r = m\angle a$$

transitive property of =

$$m\angle s = m\angle d$$

$$m\angle p = m\angle a$$

corresponding angles

$$m\angle q = m\angle d$$

$$m\angle r = m\angle p$$

transitive property of =

$$m\angle s = m\angle q$$

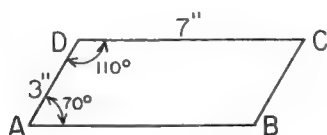
$$\angle r \cong \angle p$$

Why?

$$\angle s \cong \angle q$$

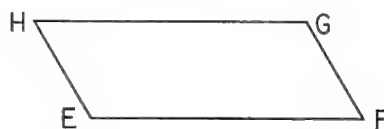
Opposite angles of a parallelogram are congruent.

Oral Figure ABCD below is a parallelogram. Use it to answer questions 1–3.



1. What is $m\angle B$? What is $m\angle C$?
2. What is $m\angle A + m\angle B$?
3. How long are \overline{AB} and \overline{BC} ?

Written Complete each sentence below about parallelogram EFGH.



1. If $m\angle E = 132$, then $m\angle G = \underline{\hspace{1cm}}$.
2. If $m\angle H = 58$, then $m\angle F = \underline{\hspace{1cm}}$.
3. $m\angle H + m\angle E = \underline{\hspace{1cm}}$.
4. $m\angle G + m\angle H = \underline{\hspace{1cm}}$.

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. Polygons are simple closed figures formed by line segments. They can be named according to the number of line segments they contain. (231)

2. Two triangles are congruent if any of the following conditions are met: (a) their corresponding sides are congruent, (b) two sides and the included angle of one are congruent to two sides and the included angle of the other, (c) two angles and the included side of one are congruent to two angles and the included side of the other. (234–237)

3. If two parallel lines are cut by a transversal, corresponding angles have the same measure. (238)

4. In a parallelogram, opposite sides are congruent and opposite angles are congruent. (244–246)

Words to Know

1. Closed figure, open figure (229)
2. Simple closed figure (230)
3. Polygon (231)
4. Scalene, isosceles, equilateral, acute, equiangular, right, and obtuse triangles (232)

5. Congruent triangles (234–237)

6. Parallel lines, transversal (238)

7. Quadrilateral, trapezoid, parallelogram, rectangle, square, rhombus (240)

Questions to Discuss

1. Is a numeral 8 a simple closed figure? Why or why not? (230)

2. If $\triangle DEF$ is an obtuse scalene triangle, what do you know about its angles and its sides? (233)

3. If figure ABCD is a square, what do you know about its angles and its sides? (240)

4. If figure PQRS is a parallelogram, what do you know about its angles and its sides? (240, 244)

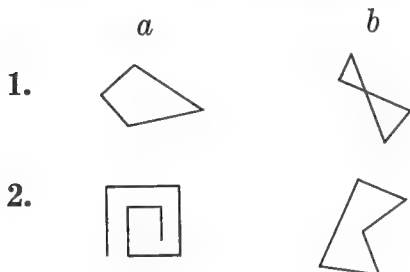
Written Practice

1. Construct a triangle, given the three sides below. (234)

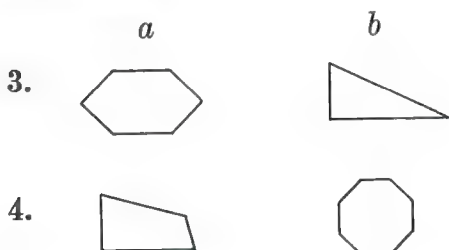
2. Construct a rectangle, given two sides as below. (242)

Self-Evaluation

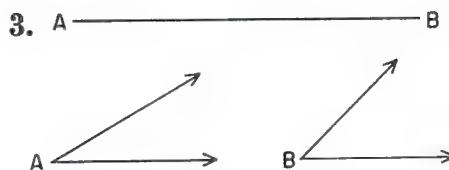
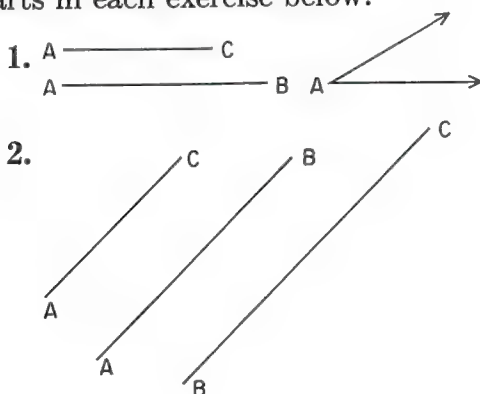
Part 1 Identify each figure below as an *open* or a *closed* figure. Then tell which are *simple closed* figures.



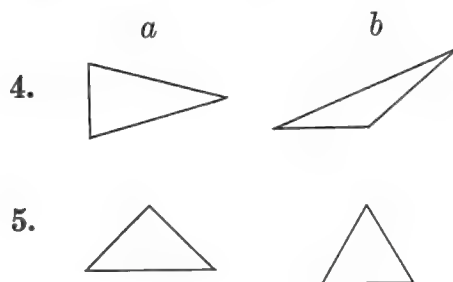
Identify each polygon below as a *triangle*, a *quadrilateral*, a *hexagon*, or an *octagon*.



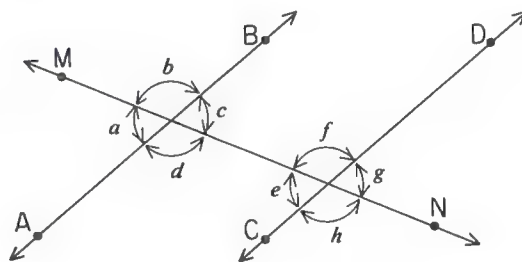
Part 2 Construct $\triangle ABC$, given the parts in each exercise below.



Give a complete classification of each triangle below.



Part 3 Use the figure below to complete each sentence below, given that $\overline{AB} \parallel \overline{CD}$.



- $\angle a$ and $\angle \underline{\hspace{1cm}}$ are corresponding angles.
- If $m\angle c = 42$, then $m\angle a = \underline{\hspace{1cm}}$.
- $m\angle a + m\angle b = \underline{\hspace{1cm}}$.
- $\angle f$ and $\angle \underline{\hspace{1cm}}$ are vertical angles.

Chapter 11

MEASUREMENT

Standard Units of Length

Different countries use different standard units of measure. The following set of standard units is commonly used in the United States and many other countries.

$$1 \text{ foot (ft.)} = 12 \text{ inches (in.)}$$

$$\frac{1}{12} \text{ ft.} = 1 \text{ in.}$$

$$1 \text{ yard (yd.)} = 3 \text{ ft.}$$

$$\frac{1}{3} \text{ yd.} = 1 \text{ ft.}$$

$$1 \text{ yd.} = 36 \text{ in.}$$

$$\frac{1}{36} \text{ yd.} = 1 \text{ in.}$$

$$1 \text{ mile (mi.)} = 5280 \text{ ft.}$$

$$\frac{1}{5280} \text{ mi.} = 1 \text{ ft.}$$

In a sentence like $1 \text{ ft.} = 12 \text{ in.}$, the $=$ means that the two measurements are two ways of expressing the same length. The measures (1 and 12) are different, the units (foot and inch) are different, but the measurements name the same length.

Explain how $4\frac{1}{4} \text{ ft.}$ is changed to 51 in. and how 108 in. is changed to 9 ft. below.

$$4\frac{1}{4} \text{ ft.} = \underline{\hspace{1cm}} \text{ in.}$$

$$1 \text{ ft.} = 12 \text{ in.}$$

$$(4\frac{1}{4} \times 1) \text{ ft.} = (4\frac{1}{4} \times 12) \text{ in.}$$

$$4\frac{1}{4} \text{ ft.} = 51 \text{ in.}$$

$$108 \text{ in.} = \underline{\hspace{1cm}} \text{ ft.}$$

$$1 \text{ in.} = \frac{1}{12} \text{ ft.}$$

$$(108 \times 1) \text{ in.} = (108 \times \frac{1}{12}) \text{ ft.}$$

$$108 \text{ in.} = 9 \text{ ft.}$$

Oral Tell how you would complete each sentence below.

a

b

1. 60 in. = ft.

6 ft. = in.

2. 9 yd. = in.

2 mi. = ft.

3. 132 in. = ft.

880 ft. = mi.

Written Complete each sentence so that it becomes true.

a

b

1. $8\frac{1}{3} \text{ yd.} = \underline{\hspace{1cm}} \text{ in.}$

42 in. = ft.

2. $5\frac{1}{2} \text{ ft.} = \underline{\hspace{1cm}} \text{ in.}$

9 yd. = ft.

3. 72 in. = yd.

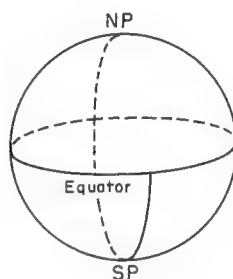
3 mi. = ft.

M P
O R A
R E C
E T I
C E
PAGE
330

Metric Units of Length

Prior to 1800, different units of measure were in common use throughout Europe. It became evident that an acceptable system of measurement should be adopted in order to facilitate manufacturing, trade, and scientific development. There was also a great desire to have a system of measurement patterned after the decimal system of numeration.

During the French Revolution (1792), the French developed the Metric System of Measurement by adopting one ten millionth ($\frac{1}{10,000,000}$) of the distance from the equator to the North Pole as the basic unit. This unit of length is called a **meter**. Larger units are 10 times, 10×10 times, $10 \times 10 \times 10$ times, etc., as great. Smaller units are $\frac{1}{10}$, $\frac{1}{10} \times \frac{1}{10}$, $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}$, etc., of a meter.



1 *megameter* = 1,000,000 meters (m.)

1 *kilometer* (km.) = 1,000 m.

1 *hectometer* (hm.) = 100 m.

1 *decameter* (dkm.) = 10 m.

1 *decimeter* (dm.) = $\frac{1}{10}$ m. or .1 m.

1 *centimeter* (cm.) = $\frac{1}{100}$ m. or .01 m.

1 *millimeter* (mm.) = $\frac{1}{1000}$ m. or .001 m.

1 *micron* = $\frac{1}{1,000,000}$ m. or .000001 m.

What is the meaning of each prefix shown in the table above? These prefixes are used with all units (such as units of weight and capacity) in the metric system.

Due to modern science, the length of a meter is now determined by using the wave length of a certain orange-red light emitted by krypton gas. The metric system was adopted in 1893 as the official system of measurement in the United States.

Explain how 24 cm. is changed to 240 mm. and how 7825 m. is changed to 7.825 km. below.

$$24 \text{ cm.} = \underline{\hspace{1cm}} \text{ mm.}$$

$$1 \text{ cm.} = 10 \text{ mm.}$$

$$(24 \times 1) \text{ cm.} = (24 \times 10) \text{ mm.}$$

$$24 \text{ cm.} = 240 \text{ mm.}$$

$$7825 \text{ m.} = \underline{\hspace{1cm}} \text{ km.}$$

$$1 \text{ m.} = \frac{1}{1000} \text{ km.}$$

$$(7825 \times 1) \text{ m.} = (7825 \times \frac{1}{1000}) \text{ km.}$$

$$7825 \text{ m.} = \frac{7825}{1000} \text{ or } 7.825 \text{ km.}$$

Oral What is the meaning of each of the following words?

a

b

1. millisecond megavolt

2. megaton milliampere

Explain how you would complete each sentence below.

a

b

3. 4 m. = cm. 2.3 km. = m.

4. 52 mm. = cm. 172 dm. = km.

Written Copy. Complete each sentence below so that it becomes true.

1. 4000 m. = km.

2. 820 cm. = m.

3. 5000 dm. = km.

4. 524 dm. = m.

5. 17900 mm. = m.

6. 78 mm. = dm.

7. 7 megameters = m.

8. 26 mm. = microns

9. 7.36 m. = cm.

10. .25 m. = dm.

11. 17 dkm. = cm.

12. 12 km. = hm.

13. 750 cm. = m.

14. 1 km. = cm.

15. 85 cm. = mm.

Tell how A meter is approximately 39.37 inches long. How can you express a length of 31 meters as a number of feet?

Can you do this? Since 1 m. = 39.37 in. approximately, then

$$1 \text{ m.} = \frac{39.37}{12} \text{ ft. (approx.)}$$

$$= 3.28 \text{ ft. (approx.)}$$

If two towns are 28 km. apart, what is the approximate distance between them in miles?

Temperature

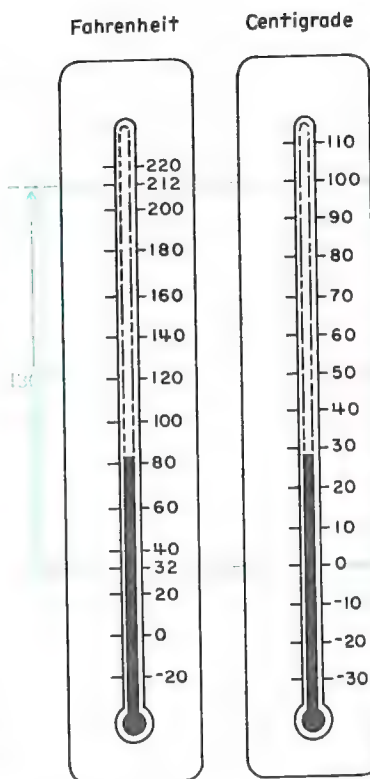
At sea level, the temperatures at which water boils and freezes are taken as fixed points on common thermometer scales. Two common scales are shown at the right.

The Fahrenheit scale is commonly used for daily purposes and weather forecasting. The centigrade scale is used in scientific work and is part of the metric system.

Between the freezing and the boiling temperatures of water, the ratio of the number of Fahrenheit units to the number of centigrade units is

$$\frac{180}{100} \text{ or } \frac{9}{5}.$$

In other words, the number of Fahrenheit units is $\frac{9}{5}$ as great as the number of centigrade units.



Notice that 32 on the Fahrenheit scale corresponds to 0 on the centigrade scale. The Fahrenheit scale has a head start of 32° over the centigrade scale. We compensate for this head start by subtracting 32 from the Fahrenheit measure as shown in the proportion below. Let F stand for the Fahrenheit measure and let C stand for the centigrade measure.

$$\frac{F-32}{C} = \frac{9}{5} \quad \rightarrow \quad 5(F-32) = 9C \quad \rightarrow \quad \begin{aligned} &F-32 = \frac{9}{5}C \quad \text{or} \quad F = \frac{9}{5}C + 32 \\ &\frac{5}{9}(F-32) = C \quad \text{or} \quad C = \frac{5}{9}(F-32) \end{aligned}$$

The two formulas shown in blue enable you to change a Fahrenheit measure to a centigrade measure or vice versa.

Explain how 113°F. (113 degrees Fahrenheit) is changed to 45°C. (45 degrees centigrade) and how 40°C. is changed to 104°F. as shown below.

$$C = \frac{5}{9}(F - 32)$$

$$C = \frac{5}{9}(113 - 32)$$

$$= \frac{5}{9}(81)$$

$$= 45$$

Thus, 113°F. = 45°C.

$$F = \frac{9}{5}C + 32$$

$$F = \frac{9}{5}(40) + 32$$

$$= 72 + 32$$

$$= 104$$

Thus, 40°C. = 104°F.

Oral Answer these questions.

1. A weather report stated that the high temperature for the day was 85°. What scale was being used? How do you know?

2. Which is the higher temperature: 70°C. or 70°F.?

3. Which is warmer, water at a temperature of 33°F. or 33°C.?

4. Which of the formulas shown above would you use to change 68°F. to a centigrade reading? Which would you use to change 115°C. to a Fahrenheit reading?

Read each of the following.

	<i>a</i>	<i>b</i>	<i>c</i>
5.	45°F.	68°C.	121°F.
6.	32°C.	-10°F.	-1°C.
7.	-17°F.	526°C.	-27°F.

Written Use the formulas above to change each temperature reading below as indicated.

	<i>Centigrade Reading</i>	<i>Fahrenheit Reading</i>
1.	115°C.	_____
2.	_____	68°F.
3.	75°C.	_____
4.	_____	95°F.
5.	15°C.	_____
6.	200°C.	_____
7.	_____	167°F.
8.	_____	140°F.

Can you do this? At what temperature will the Fahrenheit and the centigrade readings be the same?

Units of Time

Our standard units of time were originally derived from the movements of the earth as it rotates on its axis and revolves around the sun. Some units of time are now determined in a much more scientific manner by means of vibrating quartz crystals.

The following table states the various relationships among the more common units of time.

$$60 \text{ seconds (sec.)} = 1 \text{ minute (min.)}$$

$$60 \text{ min.} = 1 \text{ hour (hr.)}$$

$$24 \text{ hr.} = 1 \text{ day (da.)}$$

$$7 \text{ da.} = 1 \text{ week (wk.)}$$

$$12 \text{ months} = 1 \text{ year (yr.)}$$

Astronomers have calculated a year to be about 365 da. 5 hr. 48 min. 46 sec. Such an amount of time cannot be shown on a calendar. Therefore, a calendar shows 365 days per year except for a *leap* year which has 366 days. The extra day of a leap year is placed in the month of February.

Oral Answer these questions.

1. Which months have 30 days? 31 days? 28 or 29 days?

2. Describe a situation in which you would measure time in seconds. In minutes. In hours. In days. In years.

Written Express each time given below in the units indicated.

1. 7 min. = ____ sec.

2. 3 min. 42 sec. = ____ sec.

3. 13 yr. 10 months = ____ months

4. 5 yr. 12 da. = ____ da.

5. 4 hr. 3 min. = ____ min.

6. 14 wk. 5 da. = ____ da.

7. 2 da. 17 hr. = ____ hr.

8. 2 hr. = ____ sec.

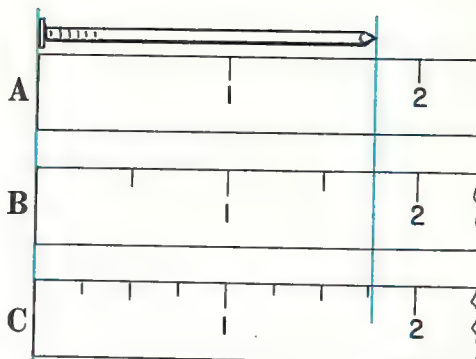
9. 321 sec. = ____ min. ____ sec.

10. 56 hr. = ____ da. ____ hr.

Measure and Measurement

The distance between two consecutive marks on ruler A is 1 inch. In this case the **unit of measure** is an inch. What is the unit of measure on ruler B? On ruler C?

By using ruler A, you can say that to the nearest unit or inch the nail is 2 inches long. The number of units is called the **measure**.



An expression like *2 inches* is called a **measurement**. A measurement consists of two symbols: a numeral to name the *measure* and a word, an abbreviation, or a symbol to name the *unit of measure*.

$\underbrace{2}_{\text{measure (number of units)}}$ $\underbrace{\text{inches}}_{\text{unit of measure}}$

Since we can read a measuring instrument only to the nearest unit into which it is graduated, any measurement is only approximate.

Oral Answer these questions about the rulers shown above.

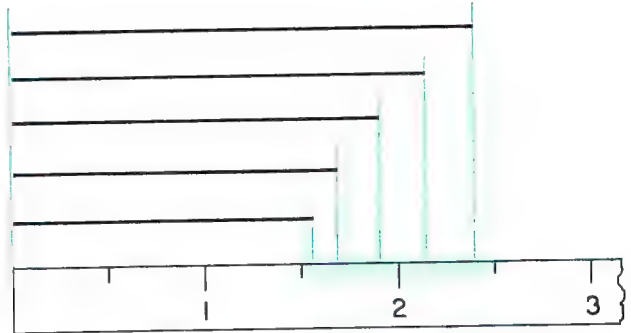
1. Using ruler B, what is the length of the nail to the nearest unit?
2. Using ruler C, what is the length of the nail to the nearest unit?
3. In the measurement " $1\frac{3}{4}$ inches," what name is given to " $1\frac{3}{4}$ "? What name is given to "inches"?
4. What is the unit in 2 inches?

Written Use a ruler to find the length of each line segment below to the nearest inch, to the nearest $\frac{1}{2}$ inch, and to the nearest $\frac{1}{4}$ inch.

1. _____
2. _____
3. _____
4. _____
5. _____

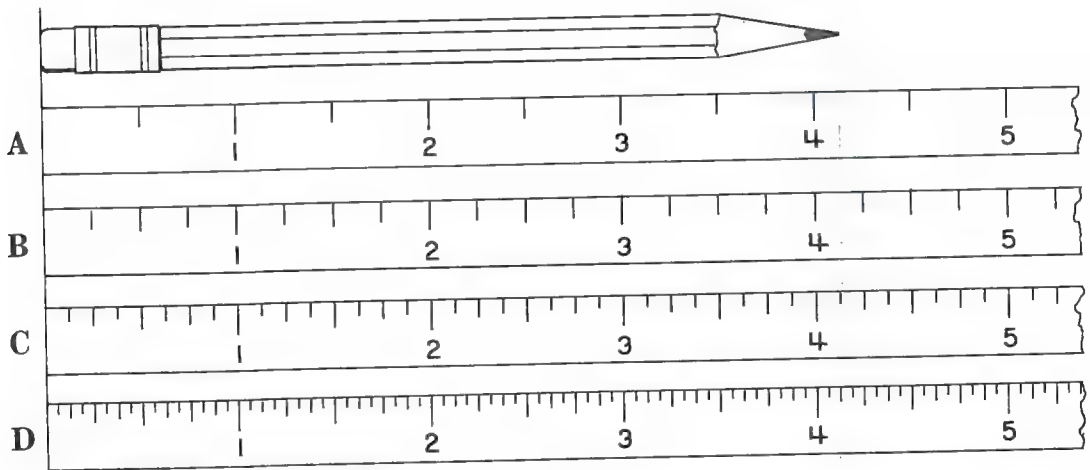
Precision

To the nearest inch, what is the length of each line segment at the right? Could you draw more line segments, each having a measurement of 2 inches to the nearest inch? The length of such a line segment can be how much less than 2 inches? How much more than 2 inches?



A measurement like *2 inches* expresses any length between $\frac{1}{2}$ unit less than 2 and $\frac{1}{2}$ unit more than 2. Hence, *2 inches* names any length between $2 - \frac{1}{2}$ or $1\frac{1}{2}$ inches and $2 + \frac{1}{2}$ or $2\frac{1}{2}$ inches. This **range of measurement** is shown in blue above.

On the rulers below, the inch unit is separated into smaller units of measure.



Using ruler A, the length of the pencil might be given as $4\frac{0}{2}$ inches. The fraction $\frac{0}{2}$ merely indicates that the unit is $\frac{1}{2}$ inch or that the measurement is made to the nearest $\frac{1}{2}$ inch. Since the unit is $\frac{1}{2}$ inch and since $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, the length of the pencil is any length between $3\frac{3}{4}$ inches and $4\frac{1}{4}$ inches.

What unit of measure is illustrated on ruler **B**? On ruler **C**? On ruler **D**? How would you express the length of the pencil by using ruler **B**? By using ruler **C**? By using ruler **D**?

Notice that the range of measurement becomes smaller as the unit becomes smaller. We say that a measurement made with ruler **B** is more **precise** than a measurement made with ruler **A** because both the unit of measure and the range of measurement are smaller.

The smaller the unit of measure, the more precise is the measurement or the greater is the **precision**.

Oral Answer questions 1–3.

1. What does the fraction $\frac{1}{4}$ indicate in the measurement $5\frac{1}{4}$ inches?

2. What is the range of measurement for the measurement $7\frac{1}{2}$ feet? For $6\frac{1}{8}$ inches?

3. If you continue to make the unit smaller and smaller, how does the range of measurement change?

Tell the unit of measure and also the range of measurement for each measurement below. Then tell which measurement in each row has the greater precision.

	<i>a</i>	<i>b</i>	<i>c</i>
4.	3 in.	$2\frac{1}{2}$ ft.	$4\frac{1}{4}$ yd.
5.	$6\frac{0}{4}$ ft.	$4\frac{1}{8}$ mi.	$17\frac{3}{4}$ yd.
6.	$9\frac{1}{2}$ in.	$8\frac{5}{16}$ in.	$1\frac{4}{5}$ mi.
7.	$11\frac{5}{10}$ ft.	$7\frac{2}{3}$ yd.	$9\frac{3}{4}$ ft.

Written For each measurement below, describe the range of measurement as follows: for $8\frac{3}{4}$ ft. write

between $8\frac{5}{8}$ ft. and $8\frac{7}{8}$ ft.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	8 in.	$3\frac{0}{2}$ ft.	$7\frac{1}{4}$ yd.
2.	$7\frac{1}{3}$ ft.	$4\frac{1}{2}$ mi.	$5\frac{3}{8}$ in.
3.	$1\frac{3}{16}$ in.	$3\frac{0}{10}$ ft.	$9\frac{3}{4}$ yd.

Find and record the length of each line segment below to the nearest $\frac{1}{4}$ inch, to the nearest $\frac{1}{8}$ inch, and then to the nearest $\frac{1}{16}$ inch.

4. _____

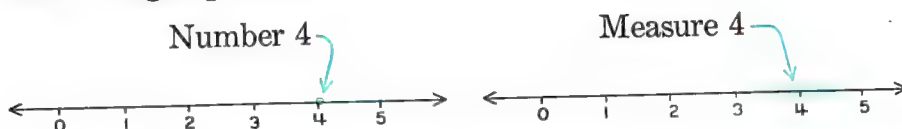
5. _____

6. _____

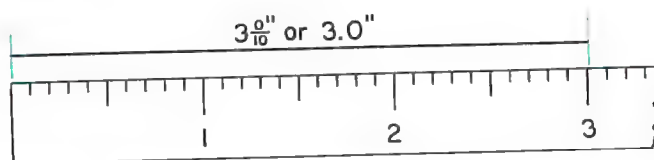
Tell why You cannot read a measurement of $67\frac{3}{5}$ in. from a ruler graduated in fourths of an inch.

Numbers and Measures

Numerals are used to name *numbers* and also to name *measures*. There is an important difference between these two uses of numerals. As shown on the number lines below, when the numeral 4 names a number it names just one number, but when the numeral 4 names a measure it refers to any number in the range $3\frac{1}{2}$ to $4\frac{1}{2}$.



In engineering and scientific work, a unit of measure is very often separated into congruent parts according to powers of $\frac{1}{10}$. This enables one to use decimals in recording or computing with measures.



The symbol " means *inch* or *inches*. The measurement $3\frac{0}{10}$ " is read $3\frac{0}{10}$ inches. How would you read 3.0"?

Just as the fraction in $3\frac{0}{10}$ " indicates a measurement to the nearest $\frac{1}{10}$ inch, so does the 0 in 3.0". The unit is .1 inch and $\frac{1}{2} \times .1 = .05$, so the length of the line segment is between $3.0 - .05$ or 2.95" and $3.0 + .05$ or 3.05".

When used to name numbers, 3.0 and 3 both name the same number. When used to name measures, 3.0 refers to a range of 2.95 to 3.05 and 3 refers to a range of 2.5 to 3.5. Hence, 3.0" and 3" name different measurements even though the numerals name the same number.

Similarly, the measurement 5.20 ft. indicates a measurement to the nearest .01 foot, while 5.2 ft. indicates a measurement to the nearest .1 foot. What is the range of measurement for 5.20 feet? For 5.2 feet?

Oral Tell which sentences below are true and which are false. If a sentence is false, tell why it is false.

- | a | b |
|----------------------------------|---|
| 1. $6.00 = 6$ | $6.00 \text{ cm.} = 6 \text{ cm.}$ |
| 2. $7\frac{1}{2} = 7.5$ | $7\frac{1}{2} \text{ in.} = 7.5 \text{ in.}$ |
| 3. $9 = 9.0$ | $9 \text{ ft.} = 9.0 \text{ ft.}$ |
| 4. $1\frac{3}{4} = 1.75$ | $1\frac{3}{4} \text{ m.} = 1.75 \text{ m.}$ |
| 5. $5\frac{1}{4} = 5\frac{2}{8}$ | $5\frac{1}{4} \text{ in.} = 5\frac{2}{8} \text{ in.}$ |

Tell the unit of measure and also the range of measurement for each of the following.

- | a | b | c |
|-------------------------------|----------------------------|-----------|
| 6. 2 yd. | $3\frac{1}{2} \text{ ft.}$ | 4.2 ft. |
| 7. $7\frac{0}{4} \text{ in.}$ | 7.0 in. | 16.23 m. |
| 8. 3.7 cm. | 1.37 m. | 27.03 mm. |

Answer the questions below.

9. John used a ruler graduated in tenths of an inch and Sue used a ruler graduated in fourths of an inch. Who can make the more precise measurement?

10. In problem 9, John found a line segment to be $7\frac{6}{10}$ inches long. He simplified the fraction and gave the measurement as $7\frac{3}{5}$ inches. How does this change the precision?

Written For each measurement below, state the range of measurement as indicated in the table. The first exercise is completed for you.

	Measure- ment	Range of Measurement
1.	26 cm.	<u>25.5 cm.</u> to <u>26.5 cm.</u>
2.	3.7 in.	_____ to _____
3.	21.0 m.	_____ to _____
4.	$2\frac{3}{4} \text{ ft.}$	_____ to _____
5.	$2\frac{6}{8} \text{ ft.}$	_____ to _____
6.	.07 cm.	_____ to _____
7.	.126 in.	_____ to _____
8.	$7\frac{7}{16} \text{ in.}$	_____ to _____
9.	20.8 m.	_____ to _____
10.	11.3 mm.	_____ to _____

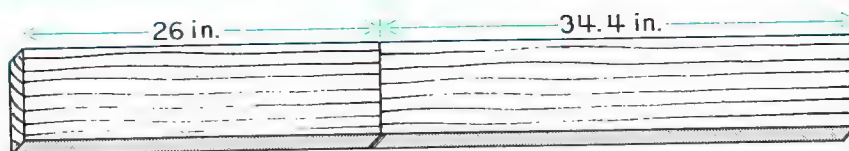
Can you do this? Instead of stating the range of measurement as in the table above, the following notation may be used.

For 5.8 cm. the range is from (5.8-.05) cm. to (5.8+.05) cm. This can be stated as

$$(5.8 \pm .05) \text{ cm.}$$

Use this notation to state the range of measurement for *Written* 1-10.

Sums and Differences of Measures



How would you record the combined length of the boards shown above? You are tempted to add the two measures.

$$\begin{array}{r} 26 \text{ in.} \\ + 34.4 \text{ in.} \\ \hline 60.4 \text{ in.} \end{array}$$

You can add only the measures. The abbreviations for units merely remind you what units the measures refer to.

This result indicates that the combined length has a range of 60.35 inches to 60.45 inches.

You know that 26 in. has a range of 25.5 in. to 26.5 in. and that 34.4 in. has a range of 34.35 in. to 34.45 in. By adding the lower and the upper limits of these ranges, you can find the range of their combined length.

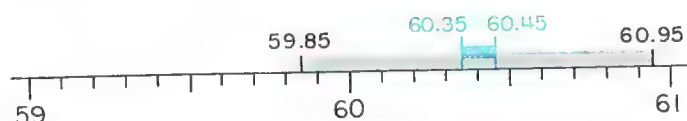
Lower Limits

$$\begin{array}{r} 25.5 \text{ in.} \\ + 34.35 \text{ in.} \\ \hline 59.85 \text{ in.} \end{array}$$

Upper Limits

$$\begin{array}{r} 26.5 \text{ in.} \\ + 34.45 \text{ in.} \\ \hline 60.95 \text{ in.} \end{array}$$

The number line below shows that the range for 60.4 in. is much too short to include all of the possible combined lengths.



The same difficulty arises when subtracting measures. To avoid these difficulties, let us make an agreement.

If two or more measurements have different precision, the sum or difference of the measures is recorded with the same precision as that of the *least precise* measurement.

For the preceding problem, record the sum of the measures as 60 in. as computed in the first example below. Notice in each example that the more precise measurement is rounded to the same precision as that of the least precise measurement before adding or subtracting.

$$\begin{array}{r} 26 \text{ in.} \\ +34.4 \text{ in.} \\ \hline \end{array} \longrightarrow \begin{array}{r} 26 \text{ in.} \\ +34 \text{ in.} \\ \hline 60 \text{ in.} \end{array}$$

$$\begin{array}{r} 3.5 \text{ ft.} \\ 2.72 \text{ ft.} \\ +4.09 \text{ ft.} \\ \hline \end{array} \longrightarrow \begin{array}{r} 3.5 \text{ ft.} \\ 2.7 \text{ ft.} \\ +4.1 \text{ ft.} \\ \hline 10.3 \text{ ft.} \end{array}$$

$$\begin{array}{r} 5.21 \text{ cm.} \\ -3.4 \text{ cm.} \\ \hline \end{array} \longrightarrow \begin{array}{r} 5.2 \text{ cm.} \\ -3.4 \text{ cm.} \\ \hline 1.8 \text{ cm.} \end{array}$$

$$\begin{array}{r} 32 \text{ m.} \\ -9.2 \text{ m.} \\ \hline \end{array} \longrightarrow \begin{array}{r} 32 \text{ m.} \\ -9 \text{ m.} \\ \hline 23 \text{ m.} \end{array}$$

You can never improve the precision of a measurement by computation. Greater precision can be obtained by using a measuring instrument graduated in the desired precision.

Oral Tell which measurement you would round off and how you would round it in order to record each sum or difference below.

- | | <i>a</i> | <i>b</i> |
|----|---|---|
| 1. | $\begin{array}{r} 17.26 \text{ in.} \\ +8.7 \text{ in.} \\ \hline \end{array}$ | $\begin{array}{r} 28.126 \text{ ft.} \\ -8.03 \text{ ft.} \\ \hline \end{array}$ |
| 2. | $\begin{array}{r} 34 \text{ mm.} \\ +7.18 \text{ mm.} \\ \hline \end{array}$ | $\begin{array}{r} 42.1 \text{ m.} \\ -15.72 \text{ m.} \\ \hline \end{array}$ |
| 3. | $\begin{array}{r} 16.8 \text{ dm.} \\ 5.26 \text{ dm.} \\ +3.6 \text{ dm.} \\ \hline \end{array}$ | $\begin{array}{r} 3.182 \text{ in.} \\ 1.07 \text{ in.} \\ +5.93 \text{ in.} \\ \hline \end{array}$ |

Compute each sum or difference below by using the agreement about precision presented in this lesson.

- | | <i>a</i> | <i>b</i> |
|----|--|---|
| 4. | $\begin{array}{r} 29.54 \text{ mi.} \\ +7.8 \text{ mi.} \\ \hline \end{array}$ | $\begin{array}{r} 26.128 \text{ in.} \\ -19.76 \text{ in.} \\ \hline \end{array}$ |
| 5. | $\begin{array}{r} 13.0 \text{ ft.} \\ +8.25 \text{ ft.} \\ \hline \end{array}$ | $\begin{array}{r} 54.0 \text{ m.} \\ -37.38 \text{ m.} \\ \hline \end{array}$ |

Can you do this? Suppose you add the measures first. Then round the result to the same precision as the least precise measurement.

$$\begin{array}{r} 26 \text{ in.} \\ +34.4 \text{ in.} \\ \hline 60.4 \text{ in.} \longrightarrow 60 \text{ in.} \end{array} \qquad \begin{array}{r} 26 \text{ in.} \\ +34 \text{ in.} \\ \hline 60 \text{ in.} \end{array}$$

Are the results the same? Use this method to do *Oral* 1–3.

Written Do the following.

1–3. Compute each sum or difference in *Oral* 1–3 by using the agreement about precision.

Adding and Subtracting Measures

Measurements may be expressed by naming more than one unit. When this is done, the precision is indicated by the smallest unit named.

11 hr. 17 min. 25 sec.	Precise to the nearest second
32 yd. 5 ft. 8 in.	Precise to the nearest inch
48 lb. 11 oz.	Precise to the nearest ounce

When recording the sum or the difference of measurements as expressed above, the least number of each kind of unit is named. Notice in the following examples that the computation is started with the numbers of the smallest unit.

$$\begin{array}{r} 5 \text{ yd.} \quad 2 \text{ ft.} \quad 10 \text{ in.} \\ + (1 \text{ yd.} \quad 2 \text{ ft.} \quad 9 \text{ in.}) \\ \hline 7 \text{ yd.} \quad 5 \text{ ft.} \quad 19 \text{ in.} \end{array}$$

7 yd. 2 ft. 7 in.

Why is 19 in. named as $(12+7)$ in.? Why is 5 ft. named as $(3+2)$ ft.?

$$\begin{array}{r} 5 \text{ m.} \quad 7 \text{ dm.} \quad 6 \text{ cm.} \\ + (8 \text{ m.} \quad 5 \text{ dm.} \quad 9 \text{ cm.}) \\ \hline 14 \text{ m.} \quad 13 \text{ dm.} \quad 15 \text{ cm.} \end{array}$$

14 m. 3 dm. 5 cm.

Why is 15 cm. named as $(10+5)$ cm.? Why is 13 dm. named as $(10+3)$ dm.?

$$\begin{array}{r} 12 \text{ gal.} \quad 1 \text{ qt.} \quad 1 \text{ pt.} \\ - (9 \text{ gal.} \quad 3 \text{ qt.} \quad 1 \text{ pt.}) \\ \hline 3 \text{ gal.} \quad 2 \text{ qt.} \quad 0 \text{ pt.} \end{array}$$

After 13 gal. is renamed, why is the number of quarts increased by 4?

$$\begin{array}{r} 10 \text{ hr.} \quad 1 \text{ min.} \quad 25 \text{ sec.} \\ - (3 \text{ hr.} \quad 20 \text{ min.} \quad 35 \text{ sec.}) \\ \hline 7 \text{ hr.} \quad 56 \text{ min.} \quad 50 \text{ sec.} \end{array}$$

After 11 hr. is renamed, why is the number of minutes increased by 60? How is 77 minutes renamed?

In the examples above, why is it unnecessary to do any rounding for precision? Why is "0 pt." retained in recording the difference 3 gal. 2 qt. 0 pt.?

Which example above is similar to the algorithms for adding and subtracting in base-ten numeration?

Oral Explain the computation for each sum below.

$$\begin{array}{r} 1. \quad 7 \text{ hr. } 13 \text{ min. } 54 \text{ sec.} \\ + (3 \text{ hr. } 48 \text{ min. } 19 \text{ sec.}) \\ \hline 11 \text{ hr. } 2 \text{ min. } 13 \text{ sec.} \end{array}$$

$$\begin{array}{r} 2. \quad 47 \text{ lb. } 13 \text{ oz.} \\ \quad 7 \text{ lb. } 3 \text{ oz.} \\ + (18 \text{ lb. } 9 \text{ oz.}) \\ \hline 73 \text{ lb. } 9 \text{ oz.} \end{array}$$

Explain the computation for each difference below.

$$\begin{array}{r} 3. \quad 3 \text{ m. } 4 \text{ dm. } 3 \text{ cm.} \\ - (1 \text{ m. } 5 \text{ dm. } 7 \text{ cm.}) \\ \hline 1 \text{ m. } 8 \text{ dm. } 6 \text{ cm.} \end{array}$$

$$\begin{array}{r} 4. \quad 7 \text{ yd. } 1 \text{ ft. } 9 \text{ in.} \\ - (4 \text{ yd. } 2 \text{ ft. } 9 \text{ in.}) \\ \hline 2 \text{ yd. } 2 \text{ ft. } 0 \text{ in.} \end{array}$$

Answer questions 5–7 about the example in *Oral* 4.

5. What is the precision of 7 yd. 1 ft. 9 in.?

6. What is the precision of 4 yd. 2 ft. 9 in.?

7. Why is the sum recorded as 2 yd. 2 ft. 0 in. instead of just 2 yd. 2 ft.?

Written Express each sum or difference by naming the least possible number of each kind of unit.

$$\begin{array}{r} 1. \quad 3 \text{ hr. } 15 \text{ min. } 42 \text{ sec.} \\ + (1 \text{ hr. } 16 \text{ min. } 24 \text{ sec.}) \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad 2 \text{ wk. } 5 \text{ da. } 17 \text{ hr.} \\ + (1 \text{ wk. } 6 \text{ da. } 21 \text{ hr.}) \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 4 \text{ yd. } 2 \text{ ft. } 8 \text{ in.} \\ \quad 3 \text{ yd. } 1 \text{ ft. } 9 \text{ in.} \\ + (5 \text{ yd. } 2 \text{ ft. } 4 \text{ in.}) \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad 18 \text{ hr. } 46 \text{ min. } 12 \text{ sec.} \\ - (6 \text{ hr. } 18 \text{ min. } 36 \text{ sec.}) \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad 7 \text{ m. } 3 \text{ dm. } 7 \text{ cm.} \\ - (4 \text{ m. } 6 \text{ dm. } 9 \text{ cm.}) \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 9 \text{ yd. } 2 \text{ ft. } 5 \text{ in.} \\ - (8 \text{ yd. } 2 \text{ ft. } 9 \text{ in.}) \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad 8 \text{ dm. } 5 \text{ cm. } 3 \text{ mm.} \\ \quad 3 \text{ dm. } 2 \text{ cm. } 6 \text{ mm.} \\ + (4 \text{ dm. } 4 \text{ cm. } 4 \text{ mm.}) \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 5 \text{ hr. } 41 \text{ min. } 19 \text{ sec.} \\ \quad 3 \text{ hr. } 16 \text{ min. } 24 \text{ sec.} \\ + (2 \text{ hr. } 37 \text{ min. } 38 \text{ sec.}) \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad 7 \text{ gal. } 2 \text{ qt. } 0 \text{ pt.} \\ - (5 \text{ gal. } 3 \text{ qt. } 1 \text{ pt.}) \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad 5 \text{ km. } 7 \text{ hm. } 3 \text{ dkm.} \\ - (1 \text{ km. } 8 \text{ hm. } 4 \text{ dkm.}) \\ \hline \end{array}$$

$$\begin{array}{r} 11. \quad 74 \text{ lb. } 7 \text{ oz.} \\ - (58 \text{ lb. } 12 \text{ oz.}) \\ \hline \end{array}$$

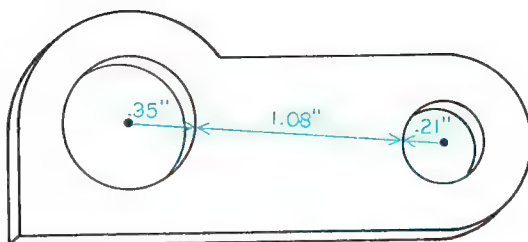
$$\begin{array}{r} 12. \quad 5 \text{ m. } 3 \text{ dm. } 7 \text{ cm. } 4 \text{ mm.} \\ \quad 8 \text{ m. } 6 \text{ dm. } 5 \text{ cm. } 8 \text{ mm.} \\ + (3 \text{ m. } 8 \text{ dm. } 2 \text{ cm. } 5 \text{ mm.}) \\ \hline \end{array}$$

$$\begin{array}{r} 13. \quad 4 \text{ mi. } 950 \text{ yd. } 2 \text{ ft. } 11 \text{ in.} \\ \quad 7 \text{ mi. } 85 \text{ yd. } 0 \text{ ft. } 7 \text{ in.} \\ + (2 \text{ mi. } 872 \text{ yd. } 1 \text{ ft. } 8 \text{ in.}) \\ \hline \end{array}$$

Equations and Computation

What is the distance between the centers of the two holes in the machine part at the right?

Units of measure are not mathematically defined. Abbreviations for units of measure are not used in open sentences.



If the measures in a problem refer to different units, change them so that all measures refer to the smallest unit. Then the measures can be referred to in an open sentence as shown below. Let d stand for the measure of the distance.

$$.35 + 1.08 + .21 = d$$

$$1.43 + .21 = d$$

$$1.64 = d \quad \text{The distance is 1.64 inches.}$$

On the other hand, any convenient symbolism may be used in algorithms. The above problem can also be solved as follows.

$$\begin{array}{r} .35 \text{ in.} \\ 1.08 \text{ in.} \\ +.21 \text{ in.} \\ \hline 1.64 \text{ in.} \end{array} \quad \text{The distance is 1.64 inches.}$$

Suppose the machine part above is .37" thick. What is the combined thickness of 15 of these parts?

To solve this problem a *measure* can be multiplied by a *number*. Let t stand for the measure of the thickness.

$$t = 15 \times .37$$

$$= 5.55 \quad \text{The combined thickness is 5.55 inches.}$$

How could you obtain this result by addition? Therefore, let us make the following agreement.

When a measure is multiplied by a number, record the product with the same precision as that of the measure.

Solve each of the following problems. Use the agreements about precision when computing with measures whenever necessary.

1. A machine can produce a plastic toy every 2.8 seconds. How long does it take the machine to produce 144 toys? (Express your answer in minutes and seconds.)

2. A circular metal tube has an inner radius of 2.185 inches and an outer radius of 2.735 inches. What is the thickness of the metal that forms the tube?

3. George is 4 ft. 6 in. tall and Sam is 57 in. tall. Who is taller? By how many inches?

4. A contractor resurfaced 2.67 miles of road one day and 3.146 miles of road the next day. How many miles of road did he resurface in the two days?

5. Assuming that water weighs about 62.5 pounds per cubic foot, what is the weight of 257 cubic feet of water?

6. After mixing 2.57 ounces of chemical with some water, the resulting solution weighed 14 lb. 6.4 ounces. What was the weight of the water?

7. The earth travels around the sun at an approximate speed of

66,700 miles per hour. How far does the earth travel in its orbit in 1 day? (Express your answer to the nearest hundred thousand miles.)

8. Each of 4 runners in a relay race ran a quarter of a mile. Their times were 58.2 seconds, 59.9 seconds, 57.4 seconds, and 60.3 seconds. What was their total time for the race?

9. The speed of sound through distilled water at 25°C . is 4913.44 feet per second and through sea water at 25°C . it travels 5021.68 feet per second. How much faster does sound travel through sea water than through distilled water?

10. An airplane took off at 10:43 A.M. and landed at 2:07 P.M. How long was the plane airborne?

Can you do this? Write an open sentence for each problem below. Solve the open sentence and answer the problem.

1. The temperature at ground level was 37°F ., but at 25,000 feet altitude the temperature was -25°F .. What is the difference between these temperatures?

2. The temperature of some water at 48°F . was increased 65°F . What was the resulting temperature of the water in centigrade?

Rate and Measurement

It is often convenient and necessary to consider two measurements that do not have the same kind of unit of measure. For example, an object may travel 24 feet in 4 seconds and you are required to state how many feet it travels in 1 second.

You can answer the above problem by using the proportion below where x stands for the number of feet the object travels in 1 second.

$$\begin{aligned}\frac{24}{4} &= \frac{x}{1} \\ 4x &= 24 \\ x &= 6\end{aligned}$$

Therefore, the object travels 6 feet in 1 second. This can be denoted by 6 ft. per sec. or 6 ft./sec. and is called a **rate**. When expressing a rate, both units of measure must be mentioned and the second measure is 1.

Oral Answer questions 1–3 below.

1. Sound travels about 1090 ft. per sec. through air (0°C.). What does 1090 ft. per sec. mean?

2. A trucking firm reported 68,000 ton-miles of business. What do you think is meant by a *ton-mile*?

3. An airline company flew 56,000 passenger-miles. What do you think is meant by a *passenger-mile*?

Written Answer each problem.

1. If 1 man works 1 hour, he performs 1 *man-hour* of work. How many *man-hours* of work are done if 9 men work for 8 hours?

2. An airplane flew 1125 miles in $2\frac{1}{2}$ hours. What was its average rate in mi. per hr.?

3. One *foot-pound* is the amount of work done in lifting 1 pound vertically through a distance of 1 foot. Bill lifted a 45-pound package 3 feet off the floor. How many foot-pounds of work was done?

4. A truck hauled 4 tons of asphalt a distance of 43 miles. How would you express this in ton-miles?

5. An airliner with 78 people aboard flew a distance of 640 miles. How would you express this in passenger-miles?

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. A sentence like 1 ft.=12 in. states that 1 ft. and 12 in. are two names for the same length. (249)

2. The metric system of measurement is patterned after the decimal system of numeration. (250)

3. You can measure only to the nearest unit indicated on the measuring instrument. (256)

4. The smaller the unit of measure, the greater is the precision of the measurement. (257)

5. If two or more measurements have different precision, the sum or difference of the measures is expressed with the same precision as that of the least precise measurement. (260)

6. The precision of measurement cannot be improved by computation, only by improved measuring. (261)

Words to Know

1. Metric system of measurement (250)

2. Fahrenheit, centigrade (252)

3. Unit of measure, measurement, measure (255)

4. Precision (257)

Questions to Discuss

1. What is the meaning of each of the prefixes: *mega*, *kilo*, *deca*, *deci*, *centi*, *milli*? (250)

2. How would you change 115°C. to Fahrenheit? 68°F. to centigrade? (253)

3. What is the precision of the measurement $5\frac{1}{2}$ in.? What is the range of measurement? (257)

4. Why is $7.0=7$ a true sentence while $7.0\text{ ft.}=7\text{ ft.}$ is a false sentence? (258)

Written Practice

Change each measurement as indicated below.

1. 60 in.=__ft. (249)

2. 7 m.=__cm. (251)

3. 65°C.=__°F. (253)

4. 104°F.=__°C. (253)

Find each sum or difference. Use the agreements about precision.

a

b

5.	17.26 m.	34.0 in.	
	<u>+8.7 m.</u>	<u>-9.26 in.</u>	(260)

Self-Evaluation

Part 1 State whether each sentence below is true or false.

1. A measurement consists of a numeral and an abbreviation, a word, or a symbol for a unit of measure.

2. The measurements $3\frac{1}{4}$ ft. and $3\frac{2}{8}$ ft. have the same precision.

3. If the length of a line segment is given as $5\frac{1}{2}$ in., its actual length is anywhere between $5\frac{1}{4}$ in. and $5\frac{3}{4}$ in.

4. A more precise measurement can be made with a ruler graduated in eighths of an inch than a ruler graduated in fourths of an inch.

5. The sum of 5.28 mi. and 2.7 mi. should be expressed to the nearest hundredth of a mile.

Part 2 Complete each sentence so that it becomes true.

- | <i>a</i> | <i>b</i> |
|--|---|
| 1. 4 yd. = ___ in. | 108 in. = ___ ft. |
| 2. 72 mm. = ___ cm. | 13 dm. = ___ cm. |
| 3. .8 m. = ___ cm. | 1200 m. = ___ km. |
| 4. $95^{\circ}\text{F.} = \text{___}^{\circ}\text{C.}$ | $15^{\circ}\text{C.} = \text{___}^{\circ}\text{F.}$ |
| 5. 7 hr. = ___ min. | 3 min. = ___ sec. |
| 6. 14 wk. = ___ da. | 8 da. = ___ hr. |

Part 3 Copy and complete the table below. The first exercise is done for you.

	<i>Measure- ment</i>	<i>Range of Measurement</i>
1.	$5\frac{1}{2}$ in.	$5\frac{1}{4}$ in. to $5\frac{3}{4}$ in.
2.	$17\frac{0}{2}$ ft.	_____ to _____
3.	17.3 m.	_____ to _____
4.	$7\frac{3}{8}$ mi.	_____ to _____
5.	576 mm.	_____ to _____
6.	$8\frac{5}{16}$ in.	_____ to _____
7.	5.38 ft.	_____ to _____
8.	.036 in.	_____ to _____

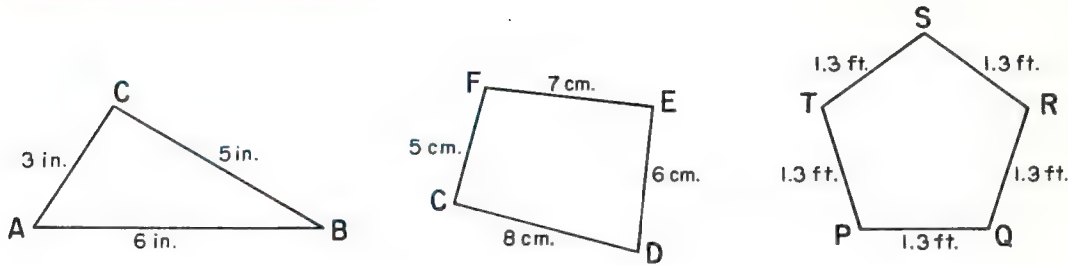
Part 4 Express each sum or difference. Use the agreements for adding and subtracting measures.

- | | <i>a</i> | <i>b</i> |
|----|--|---|
| 1. | $\begin{array}{r} 38.1 \text{ ft.} \\ + 7.35 \text{ ft.} \\ \hline \end{array}$ | $\begin{array}{r} 8 \text{ ft. } 11 \text{ in.} \\ + (5 \text{ ft. } 8 \text{ in.}) \\ \hline \end{array}$ |
| 2. | $\begin{array}{r} 72.0 \text{ m.} \\ - 8.15 \text{ m.} \\ \hline \end{array}$ | $\begin{array}{r} 17 \text{ min. } 21 \text{ sec.} \\ - (13 \text{ min. } 37 \text{ sec.}) \\ \hline \end{array}$ |
| 3. | $\begin{array}{r} 34.21 \text{ cm.} \\ 7.264 \text{ cm.} \\ + 5.8 \text{ cm.} \\ \hline \end{array}$ | $\begin{array}{r} 4 \text{ cm. } 7 \text{ mm.} \\ 5 \text{ cm. } 3 \text{ mm.} \\ + (2 \text{ cm. } 6 \text{ mm.}) \\ \hline \end{array}$ |
| 4. | $\begin{array}{r} 47.26 \text{ mi.} \\ - 34.46 \text{ mi.} \\ \hline \end{array}$ | $\begin{array}{r} 7 \text{ hr. } 13 \text{ min.} \\ - (4 \text{ hr. } 52 \text{ min.}) \\ \hline \end{array}$ |

Chapter 12

PERIMETER, AREA, VOLUME

Perimeter of a Polygon



How would you find the distance around $\triangle ABC$? That distance is called the **perimeter** of $\triangle ABC$.

If the measures of the sides of a polygon are expressed in the same unit of measurement, then the sum of the measures of the sides is the measure of the *perimeter* of the polygon.

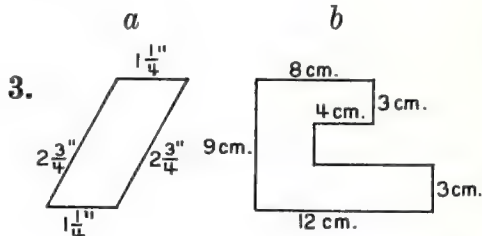
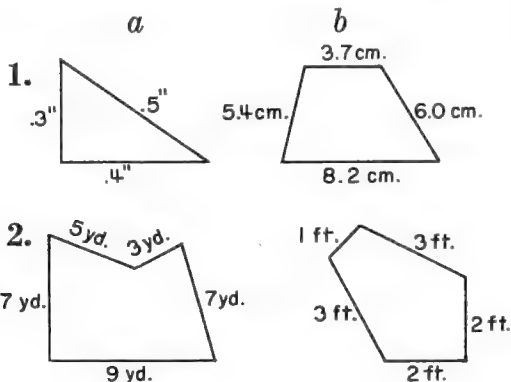
How would you find the perimeter of polygon CDEF above? Of polygon PQRST?

Oral Tell how you would find the perimeter of each polygon below.

Written Do the following.

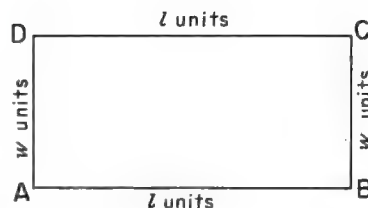
1-2. Find the perimeter of each polygon in *Oral* 1-2.

Find the perimeter of each figure below.



Formulas for Perimeter Measures

In the rectangle at the right, l and w stand for measures. Why can the lengths of both \overline{AB} and \overline{DC} be given as l units? Why can the lengths of both \overline{AD} and \overline{BC} be given as w units?

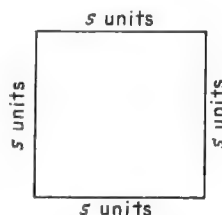


If p stands for the measure of the perimeter, a formula for the perimeter measure of any rectangle can be developed as follows.

$p = l + w + l + w$	Definition of perimeter
$= (l + l) + (w + w)$	Comm. and Assoc. Prop. of Add.
$= 2l + 2w$	Why?
$= 2(l + w)$	Why?

Formula for perimeter measure of a rectangle: $p = 2(l + w)$

If the rectangle is a square, what can you say about the measures of the 4 sides? What can you say about l and w in the above formula if the rectangle is a square? Explain in detail how the following formula for the perimeter measure of a square can be developed.



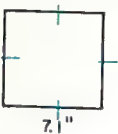
$$p = 4s$$

In $p = 4s$, what does s stand for?

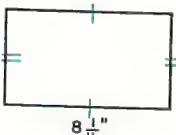
Develop a formula for the perimeter measure of each polygon shown below.




Oral Explain how the perimeter is found in each example below.

1.  $p = 4s$
 $= 4 \times 7.1$
 $= 28.4$

The perimeter is 28.4 inches.

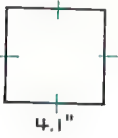

2.  $p = 2(l + w)$
 $= 2(8\frac{1}{4} + 4\frac{3}{4})$
 $= 2(13) \text{ or } 26$



The perimeter is 26 inches.

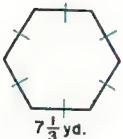

3.  $p = 6s$
 $= 6 \times 3.1$
 $= 18.6$

The perimeter is 18.6 inches.

Give open sentences like $p = 5 \times 4.7$ for finding the perimeter measure of each polygon below.

4.  a  b

5.  a  b

6.  a  b

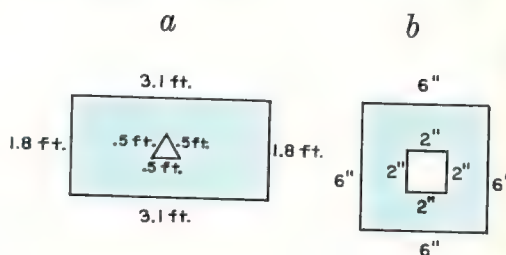
Written Find the perimeter of each polygon described below.

Type of polygon	Length of each side
1. Hexagon	$3\frac{3}{4}$ in.
2. Octagon	120 yd.
3. Pentagon	5.86 ft.

Find the perimeter of each rectangle described below.

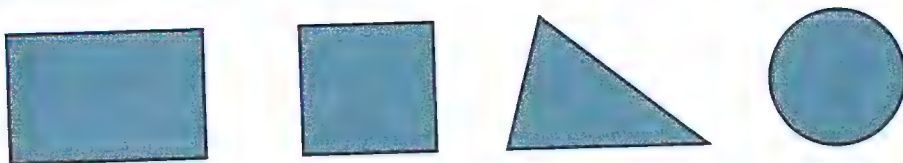
a	b
Length, Width	Length, Width
4. 5 ft., 3 ft.	1.7 mm., 1.2 mm.
5. 7 yd., 5 yd.	69 yd., 57 yd.
6. $5\frac{1}{4}$ in., $2\frac{3}{4}$ in.	11.4 mi., 4.7 mi.
7. $2\frac{3}{8}$ in., $1\frac{7}{8}$ in.	3.7 ft., 1.4 ft.
8. $4\frac{1}{4}$ in., $4\frac{1}{4}$ in.	3.7 cm., 1.5 cm.
9. $5\frac{9}{16}$ in., $2\frac{5}{16}$ in.	107 ft., 94 ft.

Tell how How would you find the overall length of the inside and outside boundaries of each colored region below?



MORE PRACTICE
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Area of a Rectangle



Each simple closed figure above encloses a definite region of a plane as its interior. Since the interior of such a figure has a definite boundary, you can measure it.

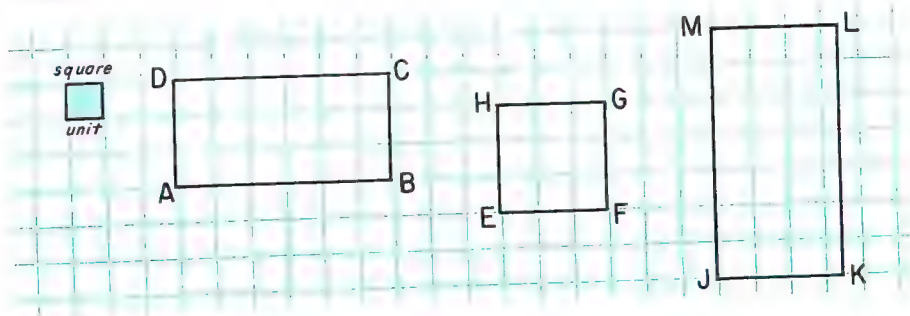
The measurement of the interior of a simple closed figure is called its **area**.

It is necessary to agree upon standard units of measurement for area. All standard units of area are derived from standard units of length. The figure chosen to define a standard unit of area is a *square*.

A square whose sides are each 1 inch long has an area of **1 square inch** or **1 sq. in.**

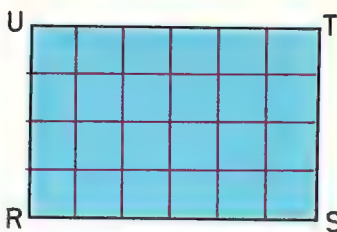
A square whose sides are each 1 foot long has an area of **1 square foot** or **1 sq. ft.**

What is the area of a square if each side is 1 mile long? 1 cm. long?



How many times is the unit of area shown above contained in the interior of each of the rectangles? For rectangle ABCD the area is 18 *square units* or 18 *sq. units*. Why is it not so easy to determine the area of rectangle JKLM?

Why is it possible to determine the area of rectangle RSTU without counting the unit squares? How can you determine the area of rectangle RSTU by using multiplication? Let l stand for the measure of the length of a rectangle, let w stand for the measure of the width of the rectangle, and let A stand for the measure of its area. Then you can use the following formula to determine the area measure of a rectangle.

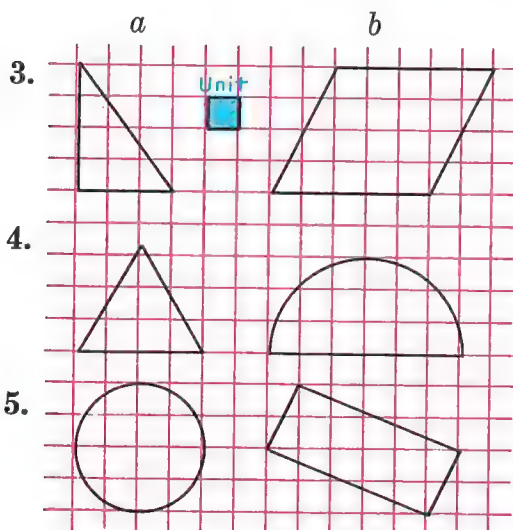


$$A = lw$$

Oral What standard unit of area is represented by a square whose sides have the following length?

- | | a | b | c |
|----|-------|-------|-------|
| 1. | 1 yd. | 1 km. | 1 dm. |
| 2. | 1 m. | 1 mm. | 1 in. |

Use the grid to estimate the area of each figure below.



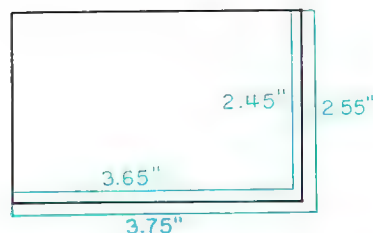
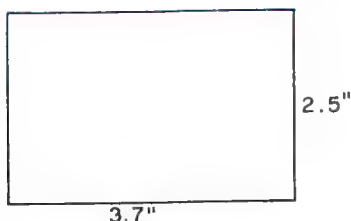
Written Use the measurements shown with each rectangle below to find its area.

- | | a | b |
|----|-----|-----|
| 1. | | |
| 2. | | |

Find the area of each rectangle described below.

- | | a | b |
|----|----------------|----------------|
| | Width, Length | Width, Length |
| 3. | 13 in., 54 in. | 16 yd., 45 yd. |
| 4. | 12 cm., 33 cm. | 24 in., 24 in. |
| 5. | 21 mi., 87 mi. | 37 ft., 20 ft. |

Computing Area Measure



The measurements of the sides of the rectangle shown above are given with a precision of $\frac{1}{10}$ or .1 inch. Hence, the length is between 3.65" and 3.75" and the width is between 2.45" and 2.55".

You can find the range of the area by first finding the product of the two minimum values of these measures and then finding the product of the two maximum values.


<i>Minimum values</i>	<i>Given measures</i>	<i>Maximum values</i>
$\begin{array}{r} 3.65 \\ \times 2.45 \\ \hline 1825 \\ 1460 \\ 730 \\ \hline 8.9425 \end{array}$	$\begin{array}{r} 3.7 \\ \times 2.5 \\ \hline 185 \\ 74 \\ \hline 9.25 \end{array}$	$\begin{array}{r} 3.75 \\ \times 2.55 \\ \hline 1875 \\ 1875 \\ 750 \\ \hline 9.5625 \end{array}$

Thus the area is between 8.9425 sq. in. and 9.5625 sq. in., but we do not know it exactly. If we reduce the precision to that of the original measurements by usual rounding off procedures, we obtain a range of 8.9 to 9.6. We are still unable to state the area exactly. This difficulty can be overcome only by an agreement.

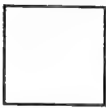
Notice that the product of the given measures is between the minimum and the maximum values of the area measure. In fact, it is very nearly midway between these values.

Until you learn more about stating products of measures, let us agree to use the product of the given measures as the area measure of a rectangle.

Oral Explain how the area of each rectangle below is found.

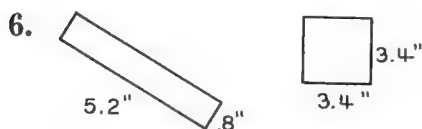
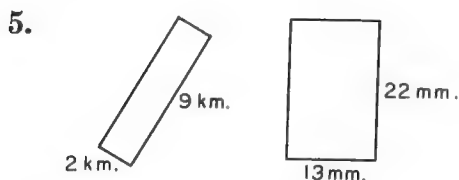
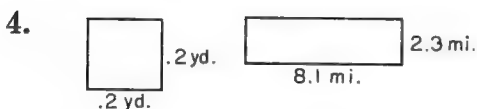
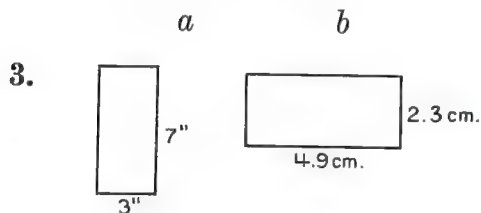
1.  $A = lw$
 $= 4.7 \times 3.5$
 $= 16.45$

Area is 16.45 sq. m.

2.  $A = lw$
 $= 3\frac{1}{2} \times 3\frac{1}{2}$
 $= 12\frac{1}{4}$

Area is $12\frac{1}{4}$ sq. ft.

Tell how you would find the area of each rectangle below. Also tell what unit of area measure you would use to express the area.



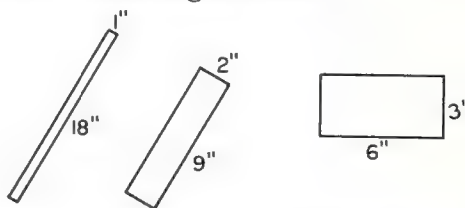
Written Find the area of each rectangle described below.

	Length	Width	Area
1.	16 ft.	13 ft.	___sq. ft.
2.	54 mm.	27 mm.	___sq. mm.
3.	84 yd.	32 yd.	___sq. yd.
4.	5.3 cm.	1.7 cm.	___sq. cm.
5.	$7\frac{3}{4}$ in.	$2\frac{1}{4}$ in.	___sq. in.
6.	8.7 m.	2.6 m.	___sq. m.
7.	$3\frac{1}{4}$ mi.	$2\frac{1}{4}$ mi.	___sq. mi.
8.	68 km.	68 km.	___sq. km.
9.	$7\frac{1}{2}$ in.	$6\frac{1}{2}$ in.	___sq. in.

Can you do this? Solve each of these problems.

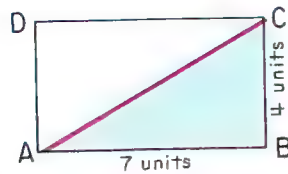
1. The area of a rectangle is 42 sq. in. and the measures of the length and width are whole numbers. Find the length and width of all the rectangles possible for this problem.

2. Find the perimeter and the area of each rectangle below.



Can different rectangles have the same area but different perimeters?

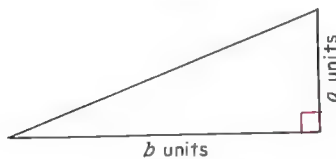
Area of a Right Triangle



Diagonal AC is drawn in rectangle ABCD at the left. What kind of angles are angles B and D? What do you know about sides AD and BC? About sides AB and CD?

Why is $\triangle ABC$ congruent to $\triangle CDA$? In geometry you will prove that congruent triangles have the same area.

The area of rectangle ABCD is (7×4) square units. Why can you say that the area of either of the right triangles is $\frac{1}{2}(7 \times 4)$ or $\frac{7 \times 4}{2}$ or 14 square units?

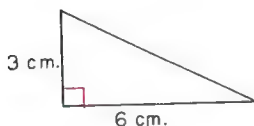


The sides that form the right angle of a right triangle can be called the *base* and the *altitude* of the triangle. Let a stand for the measure of the *altitude*, b for the

measure of the *base*, and A for the measure of the area. Then the following formula can be developed.

$$A = \frac{1}{2}ab \text{ or } A = \frac{ab}{2}$$

Observe how this formula is used to find the area of each right triangle below.



$$\begin{aligned} A &= \frac{ab}{2} \\ &= \frac{3 \times 6}{2} \\ &= 9 \end{aligned}$$

Area is 9 sq. cm.



$$\begin{aligned} A &= \frac{ab}{2} \\ &= \frac{8 \times 13}{2} \\ &= 52 \end{aligned}$$

Area is 52 sq. ft.



$$\begin{aligned} A &= \frac{ab}{2} \\ &= \frac{1.7 \times 4.2}{2} \\ &= 3.57 \end{aligned}$$

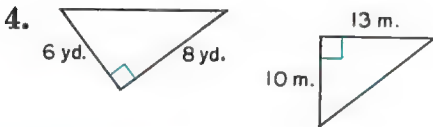
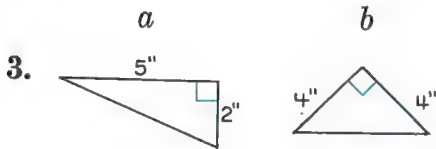
Area is 3.57 sq. in.

Oral Answer the following.

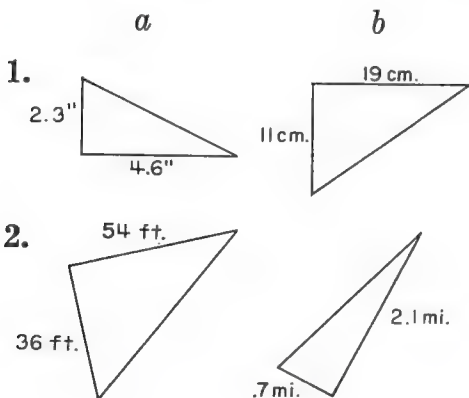
1. State in words how to find the area of a right triangle.

2. If the measurements of the sides forming the right angle of a right triangle are given in inches, what unit of area would you use in recording the area of the triangle?

Tell how you would find the area of each right triangle shown below. Then state the area of each right triangle.



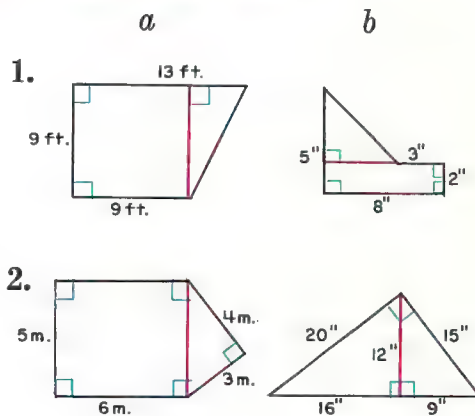
Written Find the area of each right triangle below.



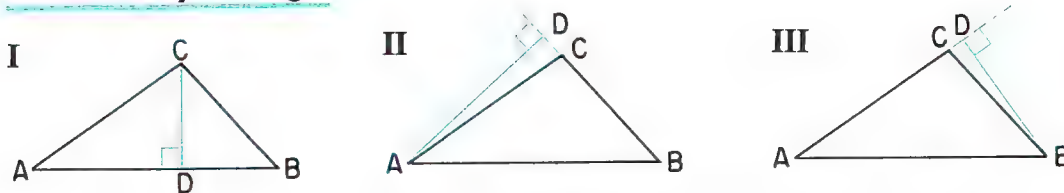
Find the area measure of each right triangle with the measurements given below.

	<i>Altitude</i>	<i>Base</i>	<i>Area</i>
3.	32 in.	12 in.	___ sq. in.
4.	8 ft.	21 ft.	___ sq. ft.
5.	8.4 m.	3.5 m.	___ sq. m.
6.	$4\frac{1}{2}$ yd.	$6\frac{1}{2}$ yd.	___ sq. yd.
7.	54 mi.	34 mi.	___ sq. mi.
8.	5.2 cm.	4.6 cm.	___ sq. cm.
9.	$5\frac{1}{4}$ ft.	$6\frac{3}{4}$ ft.	___ sq. ft.
10.	.4 km.	3.9 km.	___ sq. km.
11.	10.2 in.	26.5 in.	___ sq. in.
12.	14 yd.	14 yd.	___ sq. yd.

Can you do this? Find the area of each closed figure below.



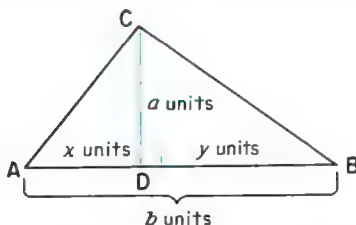
Area of Any Triangle



Each line segment shown in blue above is drawn from a vertex of $\triangle ABC$ and is perpendicular to the line that contains the side opposite that vertex. In I, \overline{CD} is called the altitude to side AB. In II, \overline{AD} is called the altitude to side BC. In III, \overline{BD} is the altitude to which side?

An *altitude* of a triangle is a line segment from any vertex perpendicular to the line that contains the opposite side.

By using the area of a right triangle, you can derive a formula for the area of any triangle. In the first case below, the altitude is in the interior of the triangle, while in the second case the altitude is in the exterior of the triangle.



The area measure of $\triangle ABC$ is the sum of the area measures of $\triangle ADC$ and $\triangle BDC$.

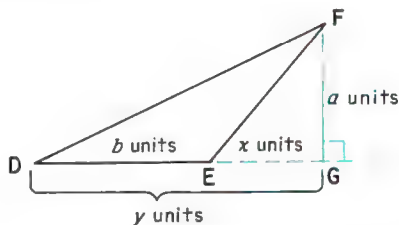
$$\text{For } \triangle ADC: A = \frac{1}{2}ax$$

$$\text{For } \triangle BDC: A = \frac{1}{2}ay$$

$$\begin{aligned} \text{For } \triangle ABC: A &= \frac{1}{2}ax + \frac{1}{2}ay \\ &= \frac{1}{2}a(x+y) \end{aligned}$$

Since $x+y=b$, replace $x+y$ by b to derive the formula

$$A = \frac{1}{2}ab.$$



The area measure of $\triangle DEF$ is the difference between the area measures of $\triangle DGF$ and $\triangle EGF$.

$$\text{For } \triangle DGF: A = \frac{1}{2}ay$$

$$\text{For } \triangle EGF: A = \frac{1}{2}ax$$

$$\begin{aligned} \text{For } \triangle DEF: A &= \frac{1}{2}ay - \frac{1}{2}ax \\ &= \frac{1}{2}a(y-x) \end{aligned}$$

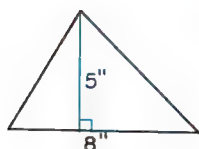
Since $y-x=b$, replace $y-x$ by b to derive the formula

$$A = \frac{1}{2}ab.$$

The formula $A = \frac{1}{2}ab$ is derived in both cases. What does a stand for in each case? What does b stand for?

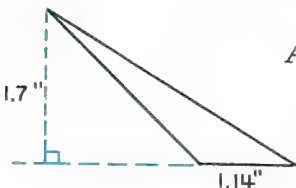
The area measure of a triangle is one half the product of the measures of a side and the altitude to that side.

Observe how the formula $A = \frac{1}{2}ab$ is used below.



$$\begin{aligned} A &= \frac{1}{2}ab \\ &= \frac{1}{2} \times 5 \times 8 \\ &= 20 \end{aligned}$$

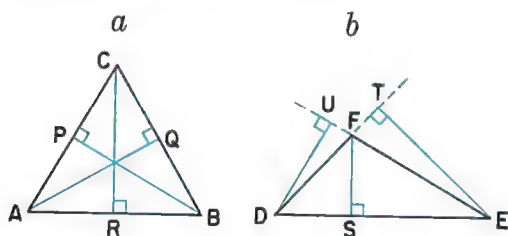
Area is 20 sq. in.



$$\begin{aligned} A &= \frac{1}{2}ab \\ &= \frac{1}{2} \times 1.7 \times 1.4 \\ &= 1.19 \end{aligned}$$

Area is 1.19 sq. in.

Oral Name the altitude to each side of each triangle below.



7. 15 mi. 18 mi. ___sq. mi.

8. 5.2 ft. 4.6 ft. ___sq. ft.

9. 6.7 km. 4.2 km. ___sq. km.

10. .8 m. 2.5 m. ___sq. m.

Find the missing measurement in each exercise below.

Written Find the area measure of each triangle described below.

	Altitude	Base	Area
1.	12 in.	9 in.	___sq. in.
2.	23 yd.	42 yd.	___sq. yd.
3.	8 cm.	6 cm.	___sq. cm.
4.	7.6 ft.	11.3 ft.	___sq. ft.
5.	36 m.	19 m.	___sq. m.
6.	$3\frac{1}{2}$ in.	$5\frac{1}{2}$ in.	___sq. in.

	Altitude	Base	Area
11.	24 yd.	___	204 sq. yd.
12.	___	14 cm.	49 sq. cm.
13.	2.4 in.	___	4.2 sq. in.
14.	___	1.7 m.	.51 sq. m.

Can you do this? Draw a large triangle and construct its 3 altitudes. Extend the altitudes until they intersect. What do you discover about their intersection?

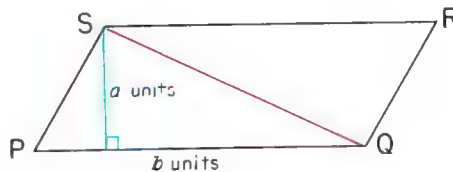
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Area of a Parallelogram



In parallelogram ABCD, line segment DE is perpendicular to the parallel sides AB and DC. Any such line segment is called an *altitude* to sides AB and CD.

You can use what you know about finding the area of a triangle to find the area of a parallelogram. Diagonal SQ is drawn in parallelogram PQRS below. What do you know about angles SPQ and SRQ? About \overline{PS} and \overline{RQ} ? About \overline{PQ} and \overline{RS} ? About $\triangle PQS$ and $\triangle RSQ$?

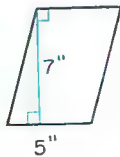


Since $\triangle PQS \cong \triangle RSQ$, their area measures are the same. Therefore, the area measure of parallelogram PQRS is the sum of the area measures of these two triangles, or twice the area measure of $\triangle PQS$.

$$\text{For } \triangle PQS: A = \frac{1}{2}ab$$

$$\begin{aligned} \text{For parallelogram PQRS: } A &= 2 \times \frac{1}{2}ab \\ &= 1ab \\ &= ab \end{aligned}$$

The area measure of a parallelogram is the product of the measures of a side and an altitude to that side.



$$\begin{aligned} A &= ab \\ &= 5 \times 7 \\ &= 35 \end{aligned}$$

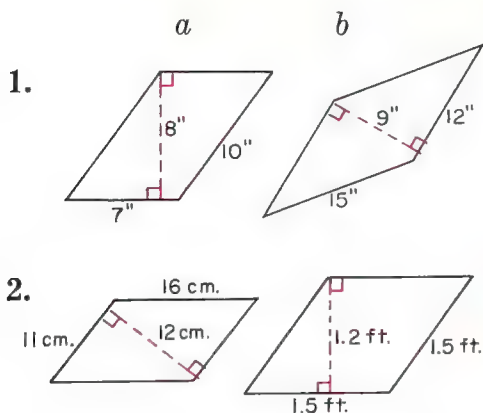
Area is 35 sq. in.



$$\begin{aligned} A &= ab \\ &= .8 \times 1.3 \\ &= 1.04 \end{aligned}$$

Area is 1.04 sq. cm.

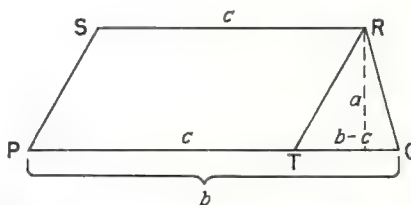
Oral Explain how you would find the area measure of each parallelogram below.



Find the area of each parallelogram described below.

a	b
Base, Altitude	Base, Altitude
3. 12 ft., 7 ft.	4 yd., 9 yd.
4. 23 in., 31 in.	9 mi., 16 mi.
5. 145 ft., 175 ft.	5.2", 6.8"
6. $3\frac{1}{2}$ yd., $2\frac{1}{2}$ yd.	6.3", 7.7"

Can you do this? In the trapezoid below, a stands for the measure of the altitude between the bases that have measures b and c .



\overline{RT} is constructed so that $\angle SRT \cong \angle SPQ$. Then PTRS is a parallelogram. Why? Therefore, $\overline{SR} \cong \overline{PT}$. Why?

The area measure of the trapezoid is the sum of the area measures of parallelogram PTRS and $\triangle TQR$.

$$\begin{aligned}
 A &= ac + \frac{1}{2}a(b-c) \\
 &= ac + \frac{1}{2}ab - \frac{1}{2}ac \\
 &= \frac{1}{2}ac + \frac{1}{2}ab \\
 &= \frac{1}{2}a(c+b) \text{ or } \frac{a(b+c)}{2}
 \end{aligned}$$

State in your own words how to find the area measure of a trapezoid.

Answer the following questions.

3. What unit of area measure would you use to record the area of the parallelogram in *Oral 1a*?

4. What unit of area measure would you use to record the area of the parallelogram in *Oral 2a*? In *Oral 2b*?

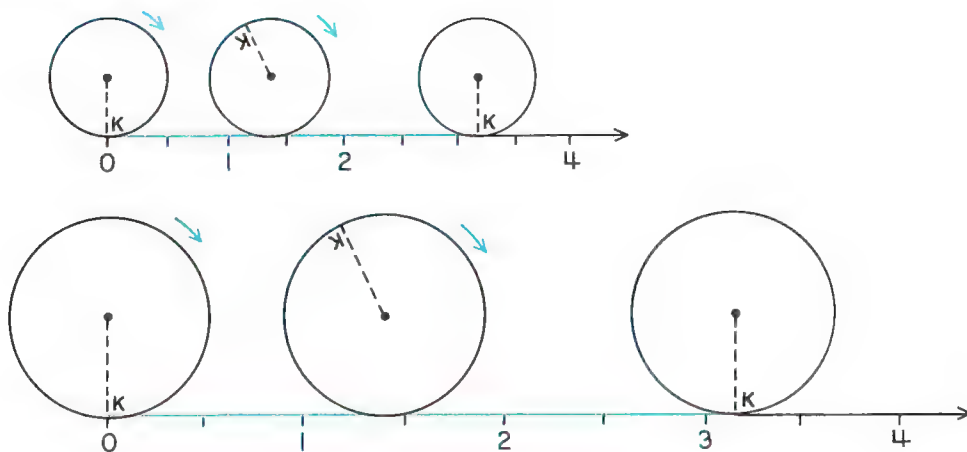
5. When the formula $A = ab$ is used to find the area measure of a parallelogram, what does each letter stand for?

6. Why can you find only an approximate area for each parallelogram shown in *Oral 1-2*?

Written Do the following.

1-2. Find the area of each parallelogram shown in *Oral 1-2*.

Circumference of a Circle



On each number ray above, the unit length is the same as the length of a diameter of the circle shown with that ray. Imagine rolling the circle along the number ray as shown above, without slipping.

The blue line segment has the same length as the **circumference** of the circle (distance around the circle). In both cases the circumference contains the length of the diameter a little more than 3 times.

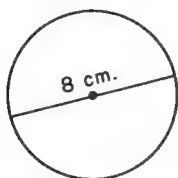
The ancient Greeks were the first to discover that this number is the same for all circles regardless of size. They used their letter π (spelled *pi*; pronounced *pie*) to denote it. If C stands for the measure of the circumference and d stands for the measure of the diameter, then

$$\frac{C}{d} = \pi \quad \text{or} \quad C = \pi d.$$

π does not name a rational number. That is, π cannot be named exactly by a fraction. Also, when π is expressed in decimal form, there is no period of digits that repeats itself.

Modern computers have calculated π to several thousand decimal places. The value of π to ten decimal places is 3.1415926536. Reasonably useful approximations of π are 3.14 and $3\frac{1}{7}$.

You can find an approximation of the circumference of a circle by using the formula $C = \pi d$ as shown below.



Using 3.14 for π

$$\begin{aligned} C &= \pi d \\ &= 3.14 \times 8 \\ &= 25.12 \end{aligned}$$

Using $3\frac{1}{7}$ for π

$$\begin{aligned} C &= \pi d \\ &= 3\frac{1}{7} \times 8 \\ &= \frac{22}{7} \times 8 \text{ or } 25\frac{1}{7} \end{aligned}$$

The circumference is approximately 25.12 cm. or $25\frac{1}{7}$ cm.

Oral Answer the following.

1. Tell two reasons why the circumference of a circle can only be expressed as an approximation.

2. How might you find the outer circumference of a bicycle tire?

3. In the formula $C = \pi d$, what does C stand for? What does d stand for?

Written Each measurement below refers to the length of a diameter of a circle. Find the circumference of each circle. Use 3.14 for π .

	a	b	c
1.	5 in.	2.5 ft.	6 ft.
2.	12 yd.	1.7 cm.	8.4 m.
3.	3 ft.	4.3 in.	3.6 yd.
4.	20 in.	3.6 in.	52 cm.
5.	17 yd.	6.6 ft.	41 mi.
6.	10 ft.	4.5 yd.	8 km.

Each measurement below refers to the length of a diameter of a circle. Find the circumference of each circle. Use $3\frac{1}{7}$ for π .

	a	b	c
7.	14 cm.	5.6 ft.	8.4 in.
8.	49 yd.	$3\frac{1}{2}$ in.	$1\frac{3}{4}$ ft.
9.	98 ft.	$7\frac{1}{2}$ cm.	3.6 in.
10.	35 mi.	$5\frac{1}{4}$ m.	7 ft.
11.	63 in.	$10\frac{1}{2}$ ft.	1.4 cm.

Can you do this? Solve each of the following problems.

1. If the circumference of a circle is 36π inches, what is the area of a square whose side is congruent to a radius of the circle?

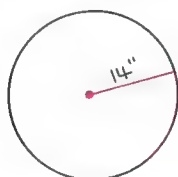
2. What is the area of a rectangle with one side congruent to a diameter and another side congruent to a radius of the circle described in problem 1?

Circumference of a Circle

A diameter of a circle is how many times as long as a radius of the same circle? Let r stand for the measure of a radius, let d stand for the measure of a diameter, and assume both measures are found by using the same unit. Then $d=2r$, and another formula for finding the circumference of a circle can be developed as follows.

$$C = \pi d \qquad C = \pi(2r) \qquad C = 2\pi r$$

$C = 2\pi r$ is particularly useful for finding the circumference of a circle when the length of a radius is given.



Using 3.14 for π

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times 3.14 \times 14 \\ &= 87.92 \end{aligned}$$

Using $3\frac{1}{7}$ for π

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times 3\frac{1}{7} \times 14 \\ &= 2 \times \frac{22}{7} \times 14 \text{ or } 88 \end{aligned}$$

The circumference is approximately 87.92 or 88 inches.

Oral Answer these questions.

1. When is $C = 2\pi r$ convenient for finding the circumference?
2. When is $C = \pi d$ convenient for finding the circumference?
3. If the length of a radius is 7 cm., what unit of measurement is used in stating the circumference?
4. The circumference of a circle is about how many times longer than a radius of the circle?
5. If a radius of a circle is 10 inches, about how long is its circumference?

Written Find the circumference of each circle whose radius or diameter has the length given below. Use 3.14 for π in exercises 1–3 and use $3\frac{1}{7}$ for π in exercises 4–6.

	Radius	Diameter
1.	9 ft.	15.1 in.
2.	3.2 cm.	8.5 yd.
3.	7.1 in.	36 cm.
4.	21 cm.	4.2 ft.
5.	$\frac{5}{7}$ mi.	$4\frac{2}{3}$ in.
6.	$8\frac{3}{4}$ yd.	$24\frac{1}{2}$ ft.

Problem Solving

Write an open sentence for each problem below. Solve the open sentence. Answer the problem.

1. Mrs. Benson has a rectangular flower garden that is 16 ft. long and 10 ft. wide. She plans to put a fence around the garden. How many feet of fence are needed? What is the area of the garden?

2. A diameter of a wheel on a bicycle is 26 in. long. How far does the bicycle travel as the wheel makes one complete revolution? Use 3.14 for π .

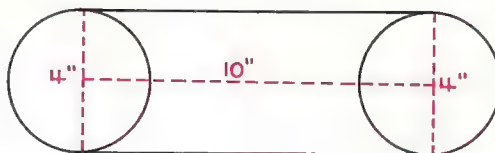
3. Mr. Johnson has a circular swimming pool with a diameter of 20 feet. He plans to erect a fence that will be 4 feet from the edge of the pool. How many feet of fence will he need? Use $3\frac{1}{7}$ for π .

4. A building lot has the shape of a right triangle. The lengths of the perpendicular sides are 125 feet and 220 feet. What is the area of the lot?

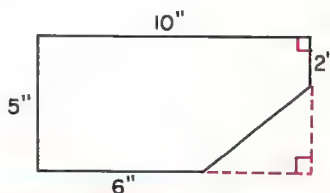
5. How many square yards of carpet are needed to cover a rectangular floor that is 24 feet by 15 feet?

6. A section of roof has the shape of a parallelogram. Its base is 18 feet long and an altitude to that base is 12 feet long. What is the area of the section of roof in square feet? In square yards?

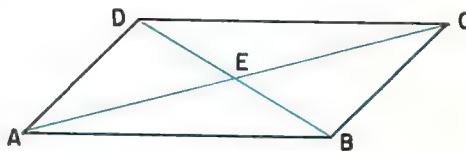
7. A belt runs around two 4-inch pulleys as shown below. How long is the belt?



8. Find the area measure of the figure below. (*Hint:* Subtract the area measure of the right triangle from the area measure of the rectangle.)



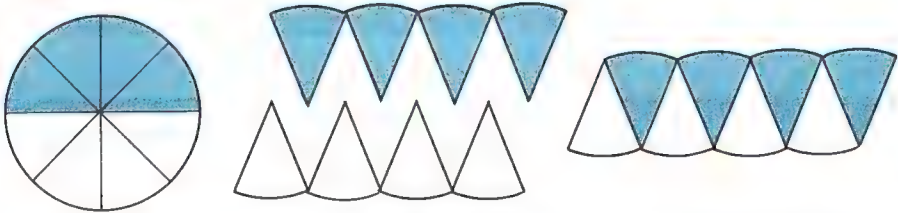
Can you do this? Both diagonals are shown in the parallelogram below. Compare the lengths of \overline{AE} and \overline{EC} . Then compare the lengths of \overline{DE} and \overline{EB} .



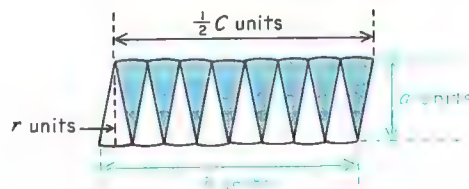
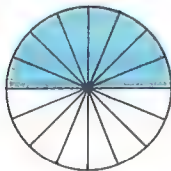
Construct several other parallelograms and repeat the above comparisons. What conclusion can you make about the diagonals of a parallelogram?

Area of a Circle

The interior of a circle can be separated into congruent parts as shown below. These parts can then be rearranged and fitted together as shown.



If the number of congruent parts becomes greater and greater, each piece becomes more and more like a triangle. The pieces can then be fitted together into an approximate parallelogram as shown below.



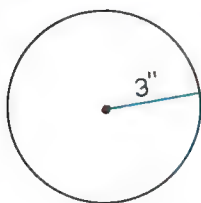
If you continue to increase the number of congruent parts of the interior of the circle, the measure of an altitude of the resulting “parallelogram” approaches the measure of a radius of the circle. The measure of a base of the “parallelogram” approaches the measure of $\frac{1}{2}$ of the circumference.

A formula for the area of a circle can be developed from the formula for the area of a parallelogram.

$$\begin{aligned}
 A &= ab \\
 &= r\left(\frac{1}{2}C\right) && \text{Replace } a \text{ by } r \text{ and } b \text{ by } \frac{1}{2}C. \\
 &= r\left(\frac{1}{2}\right)(2\pi r) && \text{Replace } C \text{ by } 2\pi r. \\
 &= r(\pi r) \\
 &= \pi r^2
 \end{aligned}$$

In more advanced mathematics it can be proved that the formula $A = \pi r^2$ gives the exact area measure of a circle.

Explain how the area of the circle is found below.



$$\begin{aligned} A &= \pi r^2 \\ &= 3.14 \times 3^2 \\ &= 3.14 \times 9 \\ &= 28.26 \end{aligned}$$

Area is 28.26 sq. in.

$$\begin{aligned} A &= \pi r^2 \\ &= 3\frac{1}{7} \times 3^2 \\ &= \frac{22}{7} \times 9 \\ &= 28\frac{2}{7} \end{aligned}$$

Area is $28\frac{2}{7}$ sq. in.

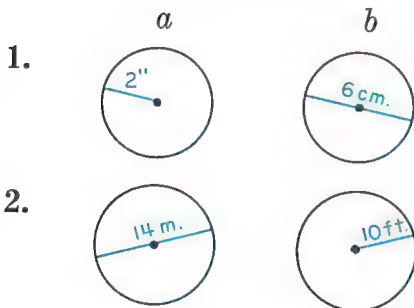
Oral Answer these questions.

1. If a radius of a circle is 5" long, what numeral would you use to replace r in $A = \pi r^2$?

2. If a diameter of a circle is 8 cm. long, what numeral would you use to replace r in $A = \pi r^2$?

3. How do you know that 28.26 and $28\frac{2}{7}$ are only approximations of the area measure above?

Written Find the area of each circle shown below. Use 3.14 for π .



Find the area of each circle described below. Use 3.14 for π .

3. Radius 5 in. _____ sq. in.

4. Radius 4 in. _____ sq. in.

5. Radius 8 ft. _____ sq. ft.

6. Radius 9 cm. _____ sq. cm.

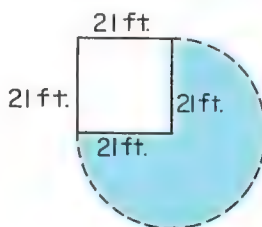
7. Diameter 12 ft. _____ sq. ft.

8. Diameter 10 yd. _____ sq. yd.

9. Diameter 24 in. _____ sq. in.

10. Diameter 30 mi. _____ sq. mi.

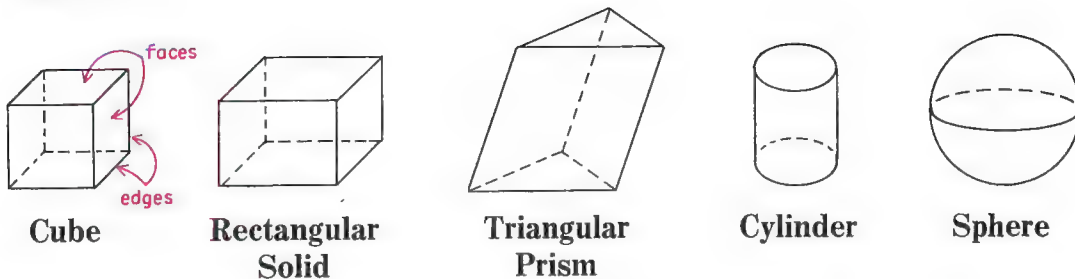
Can you do this? A horse was tied by a 21-foot rope to a corner of the square stable shown below.



How many square feet of grazing area does the horse have?

Tell how If the circumference of a circle is given as 18.84 cm. and 3.14 is used for π , how can you find the area of the circle?

Volume

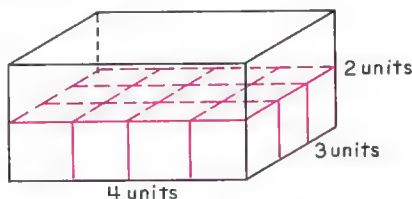


Each figure above may be called a *3-dimensional figure*, *simple closed surface*, or *solid figure*. The adjective solid does not mean they are solid like a piece of wood or iron, but merely that they enclose a portion of space.

The measurement of the interior of a solid figure is called its **volume**.

To find the volume of a solid figure, the volume of a cube is used as a standard unit. A cube has how many edges? How many faces? What is the shape of each face?

A rectangular solid is so named because each of its faces has the shape of a rectangle.

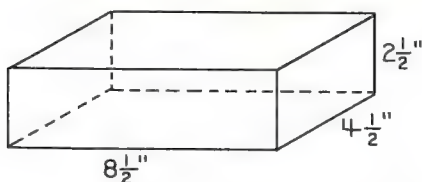


You could count the cubic units needed to fill the interior of the figure above. It is easier to obtain that number by computation. There are (4×3) cubic units in the lower layer. There would be 2 layers, so the volume is $(4 \times 3 \times 2)$ cubic units.

Let V stand for the measure of the volume and l , w , and h stand for the measures of the edges. Then the formula below can be used to find the volume measure of a rectangular solid.

$$V = lwh$$

Explain how the volume of the rectangular solid below is found by using $V = lwh$.



$$\begin{aligned} V &= lwh \\ &= 8\frac{1}{2} \times 4\frac{1}{2} \times 2\frac{1}{2} \\ &= \frac{17}{2} \times \frac{9}{2} \times \frac{5}{2} \\ &= \frac{765}{8} \text{ or } 95\frac{5}{8} \end{aligned}$$

Volume is $95\frac{5}{8}$ cu. in.

Oral If each measurement below is the length of an edge of a cube, tell the volume of the cube.

- | | a | b | c |
|----|-------|-------|-------|
| 1. | 1 in. | 1 ft. | 1 yd. |
| 2. | 1 m. | 1 cm. | 1 mi. |

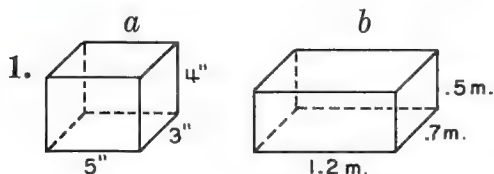
Answer the following.

3. Express in words how to find the volume of a rectangular solid.

4. If each edge of a cube is 7" long, its volume is $(7 \times 7 \times 7)$ cubic inches. How could you express $7 \times 7 \times 7$ as a power of 7?

5. If each edge of a cube is a units long, how could you express its volume measure as a power of a ?

Written Find the volume of each rectangular solid below.



Find the volume of each rectangular solid described below.

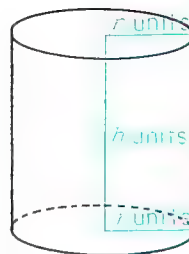
	<i>Lengths of Edges</i>	<i>Volume</i>
2.	8 yd., 4 yd., 6 yd.	___cu. yd.
3.	2.8 m., 1.7 m., .9 m.	___cu. m.
4.	$4\frac{1}{2}$ in., $3\frac{1}{2}$ in., $6\frac{1}{2}$ in.	___cu. in.
5.	12 ft., 10 ft., 8 ft.	___cu. ft.
6.	9 cm., 9 cm., 9 cm.	___cu. cm.
7.	$5\frac{1}{4}$ ", $3\frac{3}{4}$ ", $2\frac{1}{4}$ "	___cu. in.
8.	3.7 ft., 2.4 ft., 1.5 ft.	___cu. ft.
9.	15 m., 18 m., 11 m.	___cu. m.
10.	12 in., 12 in., 12 in.	___cu. in.

Can you do this? Find the missing measurements in the table below.

	<i>Lengths of Edges</i>			<i>Volume</i>
1.	___	$2\frac{1}{2}$ in.	$2\frac{1}{2}$ in.	$3\frac{1}{8}$ cu. in.
2.	1.7 ft.	___	.8 ft.	.816 cu. ft.

Volume of a Cylinder

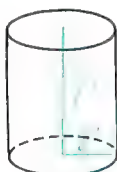
In the cylinder at the right, the top and the bottom are circles of the same size. The line joining the centers of these circles is perpendicular to every radius of each circle. Such a cylinder is called a **right circular cylinder**.



To find the volume of a cylinder it is impossible to place cubic units in a layer that completely cover the bottom. However, the number of cubic units in such a layer is the area measure of the circle. The number of layers is the measure of the height of the cylinder. Hence you can develop the following formula for the volume measure of a cylinder.

$$V = \pi r^2 h$$

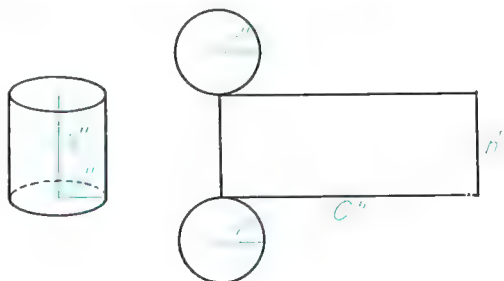
Oral Explain how the volume of the cylinder below is found.



$$\begin{aligned} V &= \pi r^2 h \\ &= 3.14 \times 2^2 \times 7 \\ &= 3.14 \times 4 \times 7 \\ &= 87.92 \end{aligned}$$

Volume is 87.92 cu. in.

Can you do this? The drawing below shows how a cylinder can be laid out flat—forming two circular regions and a rectangular region.



Written Find the volume of each cylinder described below. ($\pi = 3.14$)

	Radius	Altitude	Volume
1.	5 in.	6 in.	___cu. in.
2.	3 ft.	8.5 ft.	___cu. ft.
3.	4 cm.	3.4 cm.	___cu. cm.
4.	10 in.	34 in.	___cu. in.

Then the total area of a cylinder can be found quite easily. Add the area measures of the two circles and that of the rectangle.

Find the total area of each cylinder described in *Written* 1–4.

Checkup Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. The perimeter measure of a polygon is found by adding the measures of its sides. (269)

2. The measurement of the interior of a polygon is called the area of the polygon. (272)

3. When computing an area measure by multiplying measures, the result is only an approximation of the area measure. (274)

4. The ratio of the measure of the circumference of a circle to the measure of a diameter of that circle has the value π . (282)

5. Both 3.14 and $3\frac{1}{7}$ are approximations for π . (282)

6. The measurement of the interior of a solid figure is called its volume. (288)

Words to Know

1. Perimeter (269)

2. Area, square units (272)

3. Altitude of a triangle (278), altitude of a parallelogram (280)

4. Circumference, pi (π) (282)

5. Volume, cube, rectangular solid, cylinder (288)

Questions to Discuss

For which measurement of which figure is each formula useful?

1. $p = 2(l + w)$ (270)

2. $A = lw$ (273)

3. $A = \frac{1}{2}ab$ (276–278)

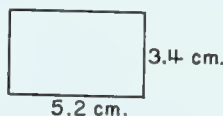
4. $C = \pi d$ (282)

5. $A = \pi r^2$ (286)

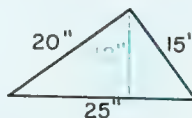
6. $V = lwh$ (288)

Written Practice

1. Find the perimeter and the area of the rectangle below. (270–275)



2. Find the perimeter and the area of the triangle below. (269, 278)



3. Find the circumference and the area of the circle below. Use 3.14 for π . (282–286)



Self-Evaluation

Part 1 Write the correct ending for each sentence below.

1. To find the measure of the perimeter of a square, multiply the measure of a side (by 2, by 4, by itself).

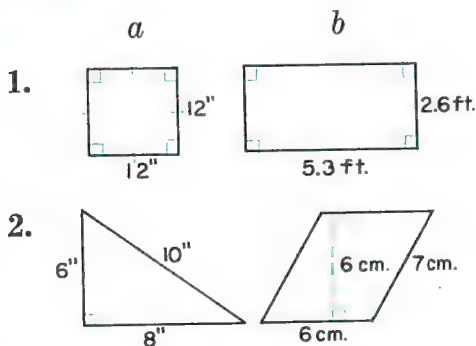
2. One altitude of a triangle is always (a side of the triangle, outside the triangle, inside the triangle).

3. For any circle, the value of π is the ratio of (d to C , C to r , C to d).

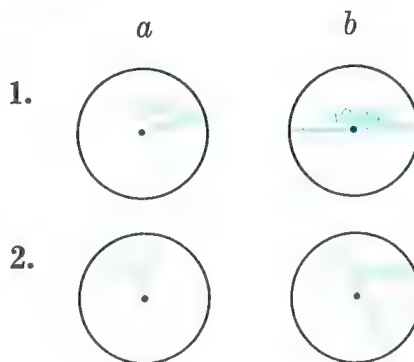
4. An altitude of a parallelogram is (the shorter side, a line segment perpendicular to opposite sides, the longer side).

5. If the length of a radius is doubled, the area of the circle becomes (twice as large, three times as large, four times as large).

Part 2 Find the perimeter and the area of each figure below.



Part 3 Find the circumference and the area of each circle below. Use 3.14 for π .

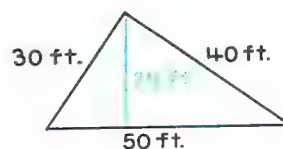


Part 4 Solve each problem.

1. A rectangular box is 12" long, 10" wide, and 8" high. How many cubic inches of sand can it contain?

2. A tin can has the shape of a circular cylinder. The radius of the top is 3 inches and its height is 4 inches. Find its volume. (Use 3.14 for π .)

3. The dimensions of a triangular garden are given below.



What is its area? How many feet of fence are needed to enclose it?

Chapter 13

ORGANIZING DATA

Sets of Data



The test scores shown above are those obtained by 10 students on a special mathematics test. From left to right they show the order in which the papers were handed in. We call such a set of scores a **set of data**.

The above set of data is *organized* since the scores are listed in the order in which the papers were handed in. What useful information might this order provide for the teacher?

Any set of data can be arranged in various ways. Various ways of organizing data will lead to various kinds of useful information about the set.

Oral Tell how you would organize the set of test scores above for each of the following purposes.

1. To enter in a grade book
2. To rank the students according to scores

Written The following is a listing of the names, heights, and weights of five boys on a sixth-grade basketball team.

Bill 67", 147 lb.
Gene 64", 135 lb.
Dan 66", 145 lb.
Ed 70", 150 lb.
Tom 62", 132 lb.

Organize this data as follows:

1. Ascending order of heights
2. Ascending order of weights
3. Alphabetical order

Organized Data

{93, 90, 85, 80, 71, 71, 68, 65, 59, 48}

The set of data above is the set of test scores on the preceding page, but organized differently. The organization is numerical from highest to lowest score. When organized in this way, much useful information about the test scores can be obtained easily.

What is the highest score? The lowest score? How many students took the test? If 70 is a passing score, how many students passed the test? How many failed? We usually express the passing or failing ratio as a per cent.

Passing

$$\frac{6}{10} = \frac{x}{100}$$

$$600 = 10x$$

$$60 = x$$

60% passed the test.

Failing

$$\frac{4}{10} = \frac{x}{100}$$

$$400 = 10x$$

$$40 = x$$

40% failed the test.

It is also possible to use the set of data to make a **ranking list** as shown below.

1st-93
2nd-90
3rd-85
4th-80
5th-71
5th-71
7th-68
8th-65
9th-59
10th-48

Notice that there are two scores of 71. Since the scores are the same, it would not be fair to rank one of them 5th and the other 6th. In such a case, the 6th place is omitted and two 5th places are assigned.

The name of the student who made each score can be substituted for that score. Then you would have the students ranked according to their scores on this test.

In the previous set of data, 71 is the only score that occurs more than once. The score of 71 is called the **mode** of the set of data.

If a score or measure occurs more frequently than any of the others in a set of data, it is called the *mode* of the set of data.

If no score occurs more than once, the set of data is said to have no mode. If different scores occur the same number of times, the set of data has more than one mode.

{6, 7, 8, 9, 14}

no mode

{51, 57, 57, 57, 64, 72, 72, 72}

two modes: 57 and 72

Oral Answer the questions following each set of data below.

{97, 93, 92, 88, 83, 78, 72, 72, 72, 64}

1. What is the mode?
 2. If 70 is a passing score, how many students passed? How many students failed?
 3. What score is ranked 5th? What score is ranked 7th?
 4. If 85 is the score for a grade of B or better, how would you find the per cent of students who will get a grade of B or better?
- {92, 86, 86, 84, 81, 78, 78, 77, 74, 71}
5. What is the mode?
 6. What rank is assigned to a score of 86?
 7. If a grade of C is given for all scores of 80 or better but less than 88, what per cent of the students will get C's?

Written Use the following set of data to do the exercises below.

{91, 83, 69, 74, 78, 57, 95, 91, 87, 64, 72, 81}

1. Organize the scores from highest to lowest score.
2. Name the mode of the set.
3. Make a ranking list of the scores.
4. If 88 is the score for a grade of B or better, find the per cent of students who will get a B or better.
5. If 80 is the score for a grade of C or better, find the per cent of students who will get a C or better and the per cent who will get less than a C.
6. If 70 is a passing score, find the per cent of students who passed and the per cent of students who failed. What is the sum of these per cents? Why?

The Median

May: {91, 91, 89, 86, 84, 83, 83, 80, 78}

June: {98, 97, 93, 90, 88, 86, 84, 82, 82, 80}

Mr. Jacobs listed his golf scores for the months of May and June. Notice that the scores are listed from highest to lowest in both cases. One way to choose a useful measure is to choose the score that occupies the middle position. That score is called the **median** of the set of data.

Notice that there is an odd number of scores for the month of May. In such cases the median is easy to determine. For the month of May the median is 84.

For the month of June there is an even number of scores. In such cases it is customary to use the average of the two middle scores as the median. The median for June is $\frac{88+86}{2}$ or 87. Observe that in this case the median is not a member of the set of data.

Oral Answer the questions below.

1. How would you find the median of {46, 54, 67, 82, 87, 89, 93}?

2. How would you find the median of {18, 26, 34, 38, 47, 56}?

3. How would you find the median of {23, 54, 85, 72, 88, 46}?

Written Find the median of each set of data. Reorganize the data if necessary.

1. {66, 72, 85, 87, 90, 97, 102}

2. {23, 27, 32, 34, 40, 47, 52, 64}

3. {114, 111, 125, 119, 118, 132, 107}

4. {1.6, 2.8, 1.7, 3.4, 2.6, 4.9, .6, 3.7}

5. {36, 43, 35, 38, 47, 42, 37, 39}

6. {.27, .15, .49, .37, .65, .07, .81}

Can you do this? Find the mode and the median of each set of data.

{18, 38, 46, 51, 51, 64, 81, 97}

{105, 38, 55, 51, 68, 55, 21, 6.3}

Can the same score be both the mode and the median of a given set of data?

The Mean

Another useful measure for Mr. Jacobs' golf scores during May is the average. The average of a set of scores is also called the **arithmetic mean** or simply the **mean** of the set.

There are 9 scores listed for the month of May. To obtain the mean, find the sum of all the scores and divide that sum by the number of scores.

$$\frac{91+91+89+86+84+83+83+80+78}{9} = \frac{765}{9} = 85$$

The mean of the set of data is 85. This means that his total score for the 9 games is the same as if he had scored 85 in each of the games. Is the mean a member of the set of data in this case?

Oral Answer the questions below.

1. Does a set of data have to be organized from lowest to highest or from highest to lowest to find the mean?

2. Do you think the mode, the median, or the mean provides the most useful information?

3. How would you find the mean of {2, 4, 5, 6, 8}? What is the mean of the set? Is the mean a member of the set?

Written Find the mean of each of the following sets of data.

1. {73, 84, 86, 96, 91, 88, 83, 69, 79, 81}

2. {1.4, 1.3, 2.4, 1.7, 1.8, 2.2, 2.5}

3. $\{1\frac{3}{4}, 2\frac{1}{4}, 2\frac{1}{8}, 1\frac{5}{8}, 1\frac{1}{2}, 2\frac{3}{8}\}$

4. {-7, -3, -2, 0, 1, 3, 6, 9, 11}

5. {169, 148, 170, 154, 161, 176}

Solve each problem below.

6. The daily high temperatures for one week were 69°, 70°, 75°, 78°, 73°, 68°, and 71°. What was the mean high temperature for that week?

7. Tom's last six bowling scores were 138, 144, 155, 142, 167, 172. What was his mean score?

8. Sue worked $2\frac{1}{2}$ hr. on Monday, $3\frac{1}{4}$ hr. on Tuesday, and $2\frac{3}{4}$ hr. on Wednesday. What is the mean number of hours worked on these three days?

Frequency

Points scored by 30 boys in a track meet

5, 4, 3, 5, 1, 6, 7, 8, 5, 3, 9, 11, 10, 6, 7,
7, 6, 6, 4, 9, 9, 2, 12, 4, 10, 7, 5, 8, 7, 6

It often happens that some scores or measures occur more than once in a set of data. The number of times each score or measure occurs in the set is called its **frequency**.

The score of 3 occurs twice in the set above; therefore its frequency is 2. What is the frequency of a score of 9?

It may be important to know the frequency of each score in a set. If so, a *frequency table* as shown below is a convenient way to arrange the data.

s	f
1 /	1
2 /	1
3 //	2
4 ///	3
5 ////	4
6 ###	5
7 ###	5
8 //	2
9 ///	3
10 //	2
11 /	1
12 /	1
Sum	30

In the frequency table, *s* stands for a score and *f* stands for its frequency. The column of tally marks is not necessary, but it is a convenient way to determine the frequency of each score.

Notice that the sum of the frequencies is the same number as the number of members in the set of data.

To find the mean of this set of data, you can compute the sum of all 30 scores, one by one, and then divide their sum by 30.

You can find the mean faster by first multiplying each score by its frequency. Then add the products to get the sum of the scores.

$$(1 \times 1) + (2 \times 1) + (3 \times 2) + (4 \times 3) + (5 \times 4) + (6 \times 5) + (7 \times 5) + (8 \times 2) + (9 \times 3) + (10 \times 2) + (11 \times 1) + (12 \times 1)$$

$$1 + 2 + 6 + 12 + 20 + 30 + 35 + 16 + 27 + 20 + 11 + 12$$

Since this sum is 192, the mean is $192 \div 30$ or 6.4.

s	f	$s \times f$
1	1	1
2	1	2
3	2	6
4	3	12
5	4	20
6	5	30
7	5	35
8	2	16
9	3	27
10	2	20
11	1	11
12	1	12
<i>Sums</i>	30	192

If you know that you will have to find the mean of a set of data, build a frequency table as shown at the left.

As soon as you make an s entry and its corresponding f entry, multiply the two numbers and record the product in the $s \times f$ -column. When all such entries are made, find the sum for the f -column and the sum for the $s \times f$ -column.

To find the mean you need only divide the sum for the $s \times f$ -column by the sum for the f -column.

What is the mode of this set of data? How can it be found from the frequency table?

What is the median of this set of data? How can it be found from the frequency table?

Oral Answer questions 1–2.

1. How can you use the f -column of a frequency table to find the mode of a set of data?

2. How can you use the f -column of a frequency table to find the median of a set of data?

Written Make a frequency table for each of the following sets of data. Then find the mode, the median, and the mean for each set of data.

1. Number of words misspelled on 20 test papers

4, 3, 1, 8, 6, 2, 4, 5, 3, 3,
2, 7, 2, 4, 5, 3, 4, 3, 6, 5

2. Number of tardinesses for 25 pupils this year

4, 5, 0, 8, 6, 2, 5, 7,
1, 3, 5, 4, 9, 6, 3, 10,
5, 4, 6, 3, 5, 8, 7, 5, 4

3. Points scored by 40 boys in a basketball tournament

5, 7, 1, 6, 10, 5, 11, 7, 3, 10,
6, 7, 8, 8, 7, 6, 13, 5, 5, 12,
3, 6, 5, 9, 2, 14, 6, 9, 7, 15,
7, 5, 8, 6, 4, 9, 7, 4, 8, 4

4. Runs scored by a baseball team in its last 30 games

3, 4, 5, 0, 8, 10, 4, 7, 3, 6,
0, 5, 1, 9, 3, 4, 11, 2, 6, 5,
4, 6, 5, 9, 5, 7, 6, 3, 4, 5

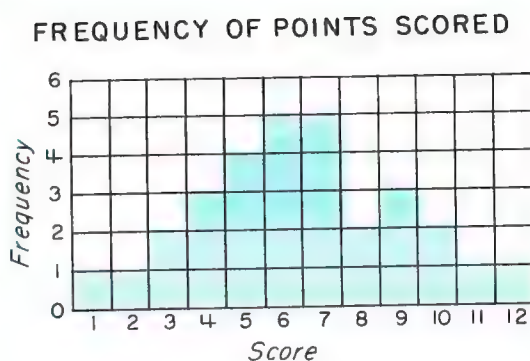
Histograms

Modes: 6 and 7

Median: 6

Mean: 6.4

<i>s</i>	<i>f</i>
1	1
2	1
3	2
4	3
5	4
6	5
7	5
8	2
9	3
10	2
11	1
12	1



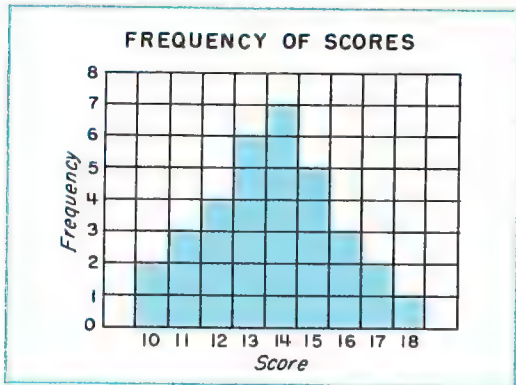
The distribution of frequencies within a set of data can be shown in a graphic, easy-to-read form as shown above. Such a graph of the frequency distribution is called a **histogram**. It is similar to a bar graph with which you are already familiar.

The scores or measures are shown on the horizontal scale and the frequencies are shown on the vertical scale. The segments on the vertical scale are congruent. The vertical scale always begins with 0.

The segments on the horizontal scale are congruent. The numerals on this scale denote the scores in the set of data. In the above case, the scores range from 1 through 12. Hence, the horizontal scale is labeled 1 through 12. Had the scores ranged from 14 through 36, the scale would have been labeled 14 through 36.

The lengths of the segments on both scales are determined by clarity, available space, and personal taste. The segments on one scale need not be congruent to the segments on the other scale.

Oral Use the following histogram to answer questions 1–4.



1. What is the mode of this set of data?

2. How many scores are in this set of data? What is the median?

3. What is your estimate of the mean for this set of data?

4. How would you compute the mean for this set of data?

Written Make a histogram for each set of data given below.

1. The enrollment in the 6th-grade classes of a city school system

Enrollment	24	25	26	27	28	29	30
Frequency	2	3	5	6	4	2	1

2. The heights in inches of the boys on two baseball teams

Height	63	64	65	66	67	68
Frequency	2	3	6	6	4	1

Find the mode, the median, and the mean of each set below. Then make a histogram for each set.

3. The daily high temperatures in degrees Fahrenheit during June

73, 70, 72, 74, 68, 75, 75, 71, 74, 73, 76, 74, 69, 75, 75, 70, 72, 76, 78, 78, 77, 77, 71, 73, 74, 79, 80, 72, 74, 75

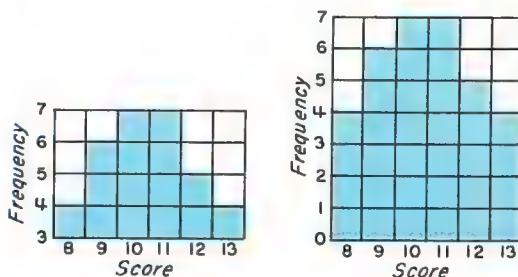
4. Number of hours of operating life of 25 flashlight batteries

20, 21, 19, 22, 18, 23, 25, 22, 23, 20, 23, 20, 22, 21, 24, 21, 22, 23, 19, 21, 22, 22, 24, 26, 22

Can you do this? Make a histogram for the frequencies of the vowels in this sentence.

Vowel	a	e	i	o	u
Frequency					

Tell why The two histograms below depict the same set of data. Compare the vertical scales.



How might the first histogram be misleading?

Probability

I have a good chance of being elected.

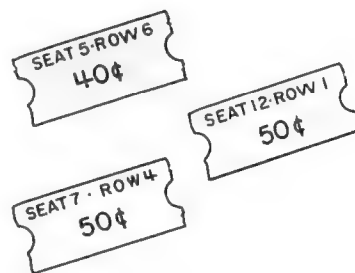
The odds are in our favor.

We will probably have a test today.

You are familiar with statements like those above. Such statements involve one of the most interesting and widely applied branches of mathematics. It is called **probability** or *probability theory*.

Insurance companies use probability to determine premium rates. Industry uses probability to control production and possible sales as well as to plan advertising programs. Politicians use it to organize election campaigns. Rules for many games are based on probability. These are but a few of its many applications.

Think of selecting one of the tickets at the right without looking. If drawing a particular ticket is called an *outcome*, how many different outcomes are possible? How many of these outcomes will produce a blue ticket? A white ticket?



If we are interested in drawing a blue ticket, let us call drawing a blue ticket a *successful outcome*. Since there are 3 possible outcomes and only 1 of these outcomes is successful (drawing a blue ticket), we say that the probability of drawing a blue ticket is 1 out of 3. We can denote "1 out of 3" by the fraction $\frac{1}{3}$.

Let n represent the number of possible outcomes of an event. Let s represent the number of successful outcomes. Then the probability of a successful outcome is $\frac{s}{n}$.

To find the probability of drawing a white ticket, $n=3$ and $s=2$. Hence, the probability of drawing a white ticket is $\frac{2}{3}$.

Oral The four selection buttons on a soft-drink machine were labeled *orange*, *grape*, *root beer*, and *cola*. Someone removed the labels so the buttons look like this:



Assume that the machine works properly. You insert a dime and push one of the selection buttons. What is the probability of getting each of the following soft drinks?

a

b

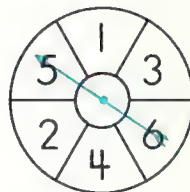
- | | |
|--------------|---------------|
| 1. orange | not orange |
| 2. grape | cola |
| 3. root beer | not root beer |

Think of selecting one marble from the box without looking.



4. What is the probability of selecting a blue marble?
5. What is the probability of not selecting a blue marble?
6. What is the sum of the probabilities in questions 4 and 5?
7. What is the probability that either a black or a blue marble will be selected?

Written Think of spinning the pointer on the dial below. Assume that it will not stop on one of the line segments. Write a fraction in simplest form for the probability of each outcome given.



1. 6
2. 1
3. an even number
4. an odd number
5. a number less than 5
6. a number greater than 1

A cup contains 4 pennies, 3 nickels, and 2 dimes. The cup is shaken until one of the coins falls out. What is the probability of each outcome below?

a

b

- | | |
|-------------|--------------|
| 7. a penny | not a penny |
| 8. a nickel | not a nickel |
| 9. a dime | not a dime |

A die is rolled once. What is the probability that the top face will show the number of dots given below?



a

b

c

- | | | |
|-------|---|-------------|
| 10. 4 | 5 | more than 0 |
| 11. 3 | 1 | less than 3 |

Different Outcomes



A lady bought 3 cans of tomatoes, 2 cans of spinach, and 5 cans of corn. Her little boy removed all the labels from the cans so that they looked alike as shown above.

If the lady opens 1 of these cans, what is the probability that it contains either spinach or corn?

One way to answer this question is as follows:

- a. There are 10 cans in all.
- b. 2 cans contain spinach and 5 cans contain corn. Hence, $5+2$ or 7 cans contain either spinach or corn.
- c. Therefore, the probability of selecting a can that contains either spinach or corn is 7 out of 10 or $\frac{7}{10}$.

Another way to answer the above question is to think of two probabilities—that of selecting a can of spinach and that of selecting a can of corn.

Probability of selecting a can of spinach: $\frac{2}{10}$ or $\frac{1}{5}$

Probability of selecting a can of corn: $\frac{5}{10}$ or $\frac{1}{2}$

Probability of selecting either a can of spinach or a can of corn:

$$\frac{1}{5} + \frac{1}{2} = \frac{2}{10} + \frac{5}{10} = \frac{2+5}{10} \quad \text{or} \quad \frac{7}{10}$$

To find the probability of selecting either a can of tomatoes or a can of corn, find the sum of the probabilities of these two separate outcomes.

$$\frac{3}{10} + \frac{2}{10} = \frac{3+2}{10} = \frac{5}{10} \quad \text{or} \quad \frac{1}{2}$$

Probability
of selecting:

tomatoes

corn

either tomatoes or corn

Oral A certain town has 3 hotels, 7 motels, and 5 cabin courts. While on vacation a family chose one of these accommodations at random. What is the probability of each of the following outcomes?

1. They stayed at a motel.
2. They stayed at a hotel.
3. They stayed at either a motel or a hotel.
4. They stayed at either a motel or a cabin court.
5. They stayed at either a hotel or a cabin court.

Written You close your eyes and strike one of the 26 letter keys on a typewriter. Find the probability that the key printed each of the following.

1. M or K
2. A vowel or the letter Z
3. J or a letter before D
4. A letter before G or a letter after W
5. A letter between H and N or a letter between Q and V
6. T or a letter in the name *John*
7. A letter in the name *Jean* or a letter in the name *Ruby*

The control panel in an elevator has a button for each of the floors 1 through 14. Pressing one of these buttons causes the elevator to proceed to that floor. You enter the elevator at the first floor and press one of the buttons without looking. The elevator works properly. Find the probability of each of these outcomes.

8. The elevator does not move.
9. The elevator goes to floor 3 or a floor above 10.
10. The elevator goes to floor 5 or an even-numbered floor.
11. The elevator goes to a floor above 7 or does not move at all.

A tire company tested 50 of its tires to determine the mileage before a defect or blowout occurred.

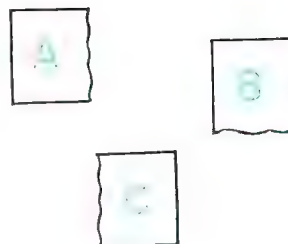
<i>Number of tires</i>	<i>Mileage</i>
8	30,000
20	20,000
10	15,000
7	10,000
5	5,000

Based on this information, find the probability of each mileage below from one of their tires.

- | | |
|----------------------|----------------|
| <i>a</i> | <i>b</i> |
| 12. less than 20,000 | 20,000 or less |
| 13. more than 10,000 | 15,000 or more |

Successive Outcomes

The slips of paper shown at the right are placed in a hat. You draw one slip of paper, note the letter on it, and replace it. The slips are then mixed and you draw again. What is the probability that you draw B both times?



We might investigate all of the possible outcomes. Let us record the letter of the first draw first and the letter of the second draw second. For example, suppose you draw A first and C second. Denote this by (A,C). Then (B,B) would denote that B was drawn both times. The set of all possible outcomes is given below.

(A,A)	(A,B)	(A,C)
(B,A)	(B,B)	(B,C)
(C,A)	(C,B)	(C,C)

How many outcomes are possible? How many of the outcomes denote that B was drawn both times? What is the probability of drawing B both times?

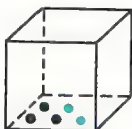
Now let us investigate the problem in another way. Consider the probabilities of the individual draws. What is the probability of B on the first draw? On the second draw?

Notice that the product of the individual probabilities is the probability of the successive outcomes.

$$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Probability of B on first draw \swarrow
 \nwarrow Probability of B on each of two successive draws
 \uparrow Probability of B on second draw

Oral Think of selecting one of the marbles from the box shown below without looking. Note its color and replace it. Mix up the marbles and again select one marble.



State each of the following probabilities.

- Both marbles are black.
- Both marbles are blue.
- The first marble is black and the second is blue.

If a coin is tossed, it must show a head (H) or a tail (T) when it comes to rest. Think of tossing the coin twice.

- What is the probability of T on the first toss? On the second toss? On both tosses?

Think of tossing two coins at the same time. The table below shows the possible outcomes.

		Second coin	
		H	T
First coin	H	HH	HT
	T	TH	TT

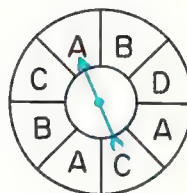
- What is the probability of two T's? Is this the same probability as for question 4?

- What is the probability of a head and a tail (disregarding their order)?

- How does tossing two coins at the same time compare with tossing a single coin twice?

Written Think of making two spins of the pointer below. What is the probability of the successive outcomes given?

- First A, then C
- First B, then D
- B both times
- A both times
- First C or D, then A



Complete the table below to show the sum of the numbers of dots on the top faces when two dice are rolled at the same time.

		Second die					
		1	2	3	4	5	6
First die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4					

Use the table to find the probability of each sum below.

- | | | | |
|----|----------|----------|----------------|
| | <i>a</i> | <i>b</i> | <i>c</i> |
| 6. | 7 | 4 | greater than 9 |
| 7. | 2 | 6 | less than 7 |

Successive Outcomes

Think of placing the tickets at the right in a bowl. Draw out one ticket without looking. *Do not replace that ticket.* Then draw another ticket. What is the probability that both tickets are blue?



One way to determine the probability is to make a listing of all the possible outcomes and then count the successful outcomes. Let w_1 and w_2 denote the white tickets, let b_1 , b_2 , and b_3 denote the blue tickets, and let g_1 and g_2 denote the gray tickets. The complete listing is as follows.

(w_1, w_2)	(w_2, w_1)	(b_1, w_1)	(b_2, w_1)	(b_3, w_1)	(g_1, w_1)	(g_2, w_1)
(w_1, b_1)	(w_2, b_1)	(b_1, w_2)	(b_2, w_2)	(b_3, w_2)	(g_1, w_2)	(g_2, w_2)
(w_1, b_2)	(w_2, b_2)	(b_1, b_2)	(b_2, b_1)	(b_3, b_1)	(g_1, b_1)	(g_2, b_1)
(w_1, b_3)	(w_2, b_3)	(b_1, b_3)	(b_2, b_3)	(b_3, b_2)	(g_1, b_2)	(g_2, b_2)
(w_1, g_1)	(w_2, g_1)	(b_1, g_1)	(b_2, g_1)	(b_3, g_1)	(g_1, b_3)	(g_2, b_3)
(w_1, g_2)	(w_2, g_2)	(b_1, g_2)	(b_2, g_2)	(b_3, g_2)	(g_1, g_2)	(g_2, g_1)

Observe that there are 42 possible outcomes. The 6 outcomes encircled above contain 2 blue tickets. Hence, the probability of both tickets being blue is $\frac{6}{42}$ or $\frac{1}{7}$.

Another way to determine the probability that both tickets are blue is to determine the individual probabilities. The probability of a blue ticket on the first draw is 3 out of 7 or $\frac{3}{7}$. Then how many tickets remain in the bowl? How many of the remaining tickets are blue? The probability that the second ticket is blue is 2 out of 6 or $\frac{2}{6}$ or $\frac{1}{3}$.

The probability that both tickets are blue is the product of the individual probabilities.

$$\frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$$

Oral You are blindfolded and select two cards in succession without replacement from the cards shown below.



1. What is the probability of selecting an A card on the first draw?

2. What is the probability of selecting another A card on the second draw?

3. How would you find the probability of selecting two A cards in succession?

4. How would you find the probability of selecting an A card first and an E card second?

5. How would you find the probability of both cards being labeled with a vowel?

Written Mary selected one of the hats below. The hat was not replaced. Then Bill selected one of the remaining hats.



Find the probability of each of the following outcomes.

1. Mary blue, Bill blue
2. Mary white, Bill gray
3. Mary white, Bill white
4. Mary blue, Bill gray

You buy 4 tickets out of 100 tickets in a raffle. The first two tickets drawn win prizes. Find the probability of each outcome below.

5. You win, then lose.
6. You win, then win again.
7. You lose, then win.

Can you do this? From the box shown below you are to select 3 marbles in succession without replacing any of them.

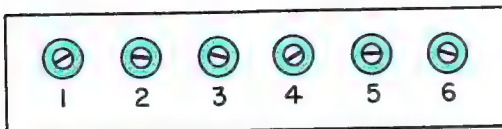


Find the probability of selecting the marbles in each order below.

1. Blue, blue, blue
2. Blue, white, white
3. Black, black, white
4. White, white, blue

Practice with Probability

Six of the fuses in a switch box appeared as follows.



Someone removed the card telling which circuit was controlled by each fuse. However, there is one fuse for each of the following:

furnace bathroom dining room
dryer kitchen living room

1. You remove one fuse. What is the probability that you break the kitchen circuit?

2. You remove one fuse. What is the probability that you break either the furnace circuit or the dryer circuit?

3. You remove one fuse. Replace it. Later another person removes one fuse. What is the probability that both of you break the bathroom circuit?

4. You remove two fuses in succession without replacing the first. What is the probability that you break the living room circuit first and the furnace circuit second?

5. You remove one fuse. What is the probability that you break one of the six circuits?

To determine what role each of 30 pupils will have in a play, each pupil draws one of the slips described below. Once a slip is drawn it is not replaced.

<i>Number of slips</i>	<i>Labeled</i>
5	clown
10	soldier
8	dancer
7	guard

6. You draw the first slip. What is the probability that you will be a dancer?

7. You draw first and Mary draws second. What is the probability that both of you will be soldiers?

8. You draw first. What is the probability that you will be either a dancer or a guard?

9. Two pupils have drawn and you are third to draw. What is the probability that all three of you will be soldiers?

10. In problem 9, what is the probability that the first two pupils will be soldiers and you will be a clown?

11. The first 10 pupils each drew a slip labeled *soldier*. You draw next. What is the probability that you will be a clown?

Checkout Time

The numerals in () tell the pages where you can turn for help.

Important Ideas

1. A set of data can be arranged in various ways. (293)
2. A set of data may have no mode, one mode, or more than one mode. (295)
3. The median and the mean may or may not be a member of the set of data. (296–297)
4. The probability that either one or the other of two different outcomes occurs in a single trial is the sum of their probabilities. (304)
5. The probability that two outcomes will occur in succession is the product of their probabilities. (306–309)

Words to Know

1. Set of data (293)
2. Ranking list (294)
3. Mode (295)
4. Median (296)
5. Mean (297)
6. Frequency (298)
7. Histogram (300)
8. Probability (302)

Questions to Discuss

1. When will a set of data have no mode? Two modes? (295)
2. When is the median not a member of the set of data? (296)
3. How do you find the mean of a set of data? (297)
4. How do you determine the height of each bar when making a histogram? (300)
5. What does a probability of $\frac{3}{7}$ mean? (302)

Written Practice

1. Find the mode, the median, and the mean for the set of data below. Then make a histogram for the set of data. (295–297, 300)

Number of fiction books read by 28 sixth-grade pupils

4, 6, 1, 7, 3, 9, 5,
7, 8, 5, 4, 6, 10, 6,
9, 5, 6, 6, 8, 6, 8,
5, 10, 7, 2, 5, 3, 7

2. On one roll of a die, what is the probability that the top face shows 4? Shows a number less than 4? (302)

Self Evaluation

Part 1 The test scores of 8 pupils are ranked below.

1st	2nd	3rd	3rd	5th	6th	7th	8th
93	85	83	83	79	78	73	61

Write an answer for each question below.

1. What is the mode of this set of data?

2. Why are two of the scores ranked 3rd?

3. If 75 is the passing score for this test, what per cent of the pupils passed? What per cent failed?

4. If 80 is the score for a grade of C or better, what per cent of the pupils will get a grade of C or better?

Part 2 Find the mode, the median, and the mean for each set of data below.

1. {4, 6, 8, 8, 9, 10, 11}

2. {2, 2, 7, 9, 11, 15, 21, 21}

3. {34, 23, 52, 85, 71, 23}

4. $\{7\frac{1}{2}, 3\frac{1}{4}, 1\frac{5}{8}, 5\frac{3}{4}, 2\frac{3}{8}\}$

5. { .2, .9, 1.1, .2, .2, 1.4, .9 }

6. {46, 40, 44, 48, 42, 44}

Part 3 Find the mode, the median, and the mean for each set of data below. Then make a histogram for each set of data.

1. Attendance at 30 club meetings this year

10, 11, 12, 8, 12, 13, 14, 11, 11, 13, 13, 12, 12, 12, 9, 14, 11, 15, 13, 12, 11, 13, 13, 11, 12, 13, 10, 14, 15, 10

2. Number of words in 24 lines of type

11, 10, 12, 12, 11, 13, 12, 11, 12, 12, 14, 10, 13, 13, 13, 12, 13, 12, 14, 12, 10, 13, 11, 12

Part 4 You are to select marbles from the box below without looking.



1. What is the probability of selecting a black marble on the first selection?

2. On the first selection, what is the probability that you get either a white or a blue marble?

3. You select a marble and do not replace it. Then you select another marble. What is the probability of the first being blue and the second being white?

Chapter 14

REVIEW EXERCISES

This chapter contains a review of many important ideas you have used. Study the statement and the example in each colored region before doing the exercises that follow it.

Numeration Systems

Egyptian numeration is additive.

$$\textcircled{9} \textcircled{1} \textcircled{1} / = 100 + 10 + 10 + 1 = 121$$

Roman numeration is additive and has a subtractive pattern.

$$\text{XXVI} = 10 + 10 + 5 + 1 = 26$$

$$\text{XLIX} = (50 - 10) + (10 - 1) = 49$$

Use expanded notation to change a base-five numeral to a base-ten numeral.

$$\begin{aligned} 214_{\text{five}} &= (2 \times 5^2) + (1 \times 5^1) + (4 \times 5^0) \\ &= (2 \times 25) + (1 \times 5) + (4 \times 1) \\ &= 50 + 5 + 4 = 59 \end{aligned}$$

Use after page 9.

Write a decimal numeral for each Egyptian or Roman numeral below.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\textcircled{1} \textcircled{1} //$	$\textcircled{9} \textcircled{1} ///$	$\textcircled{1} \textcircled{9} \textcircled{1} /$
2. CCL	XXVIII	MDLX
3. CDV	XLIV	MDXXIV
4. XLVI	MCML	LVII

Write a Roman numeral for each decimal numeral below.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
5. 14	436	629	993
6. 57	497	2563	1968

Use after page 17.

Change each base-five numeral below to the simplest base-ten numeral.

<i>a</i>	<i>b</i>	<i>c</i>
1. 13_{five}	423_{five}	2000_{five}
2. 22_{five}	332_{five}	3134_{five}
3. 40_{five}	241_{five}	4223_{five}
4. 31_{five}	104_{five}	1312_{five}
5. 44_{five}	310_{five}	2441_{five}
6. 30_{five}	413_{five}	1333_{five}
7. 14_{five}	244_{five}	2040_{five}
8. 23_{five}	141_{five}	3134_{five}

Numeration Systems

Expanded notation aids in changing a numeral in any base to a base-ten numeral.

$$\begin{aligned} 213_{\text{seven}} &= (2 \times 7^2) + (1 \times 7^1) + (3 \times 7^0) \\ &= (2 \times 49) + (1 \times 7) + (3 \times 1) \\ &= 98 + 7 + 3 = 108 \end{aligned}$$

$$\begin{aligned} 101_{\text{two}} &= (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) \\ &= (1 \times 4) + (0 \times 2) + (1 \times 1) \\ &= 4 + 0 + 1 = 5 \end{aligned}$$

Use after page 21.

Change each numeral below to a base-ten numeral.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	45_{seven}	215_{seven}	602_{seven}
2.	10_{seven}	100_{seven}	200_{seven}
3.	36_{seven}	412_{seven}	506_{seven}
4.	63_{seven}	142_{seven}	340_{seven}
5.	55_{seven}	333_{seven}	245_{seven}
6.	46_{seven}	261_{seven}	642_{seven}
7.	11_{two}	100_{two}	101_{two}
8.	10_{two}	110_{two}	111_{two}
9.	1100_{two}	1010_{two}	1111_{two}
10.	10000_{two}	10001_{two}	11001_{two}
11.	1011_{two}	1101_{two}	1001_{two}

Dividing by powers of the new base number enables you to change a base-ten numeral to a numeral in any other base.

$$\begin{array}{r} 5^0 = 1 \quad \underline{25} \overline{) 82} \quad \underline{3} \\ 75 \\ 5 \\ 5^1 = 5 \quad \underline{5} \overline{) 7} \quad \underline{1} \\ 5 \\ 2 \\ 5^2 = 25 \quad \underline{1} \overline{) 2} \quad \underline{2} \\ 2 \\ 0 \\ 5^3 = 125 \end{array}$$

$$82 = 312_{\text{five}}$$

Use after page 25.

Change each base-ten numeral below to a base-five numeral.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
1.	17	49	120	204
2.	26	87	147	356

Change each base-ten numeral below to a base-seven numeral.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
3.	19	50	154	413
4.	43	67	207	608

Change each base-ten numeral below to a base-two numeral.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
5.	7	11	16	23
6.	8	13	18	32

Mathematical Sentences

() and [] are used as grouping symbols in mathematical sentences.

$$(5 \times 6) + 4 = 30 + 4 = 34$$

$$5 \times (6 + 4) = 5 \times 10 = 50$$

Treat the innermost grouping symbols first.

$$\begin{aligned} 2 \times [3 + (7 - 5)] &= 2 \times [3 + 2] \\ &= 2 \times 5 \\ &= 10 \end{aligned}$$

Use after page 31.

Write the simplest decimal numeral for each expression below.

a

b

1. $4 \times (8 \div 2)$ $5 + [(4 \times 7) - 3]$

2. $(4 \times 8) \div 2$ $5 + [4 \times (7 - 3)]$

3. $(12 \div 3) - 2$ $(5 + 4) \times (7 - 3)$

4. $12 \div (3 - 2)$ $[(5 + 4) \times 7] - 3$

5. $(18 + 6) \div 3$ $[5 + (4 \times 7)] - 3$

6. $18 + (6 \div 3)$ $3 \times [(5 \times 3) - (3 + 4)]$

Insert grouping symbols so each expression becomes a name for the number indicated after it.

7. $64 - 14 \div 2 + 5$ Number: 52

8. $64 - 14 \div 2 + 5$ Number: 30

9. $64 - 14 \div 2 + 5$ Number: 62

A closed sentence is either true or false. An open sentence is neither true nor false. An equation contains an = and an inequality contains either <, >, or \neq .

$3 \times 4 = n$ Open sentence

$7 - 2 > 1$ True closed inequality

$8 \div 3 = 4$ False closed equation

Use after page 37.

As shown above, identify each mathematical sentence below as an open or closed equation or inequality. Also, if a sentence is closed, identify it as true or false.

a

b

1. $25 \div a = 5$ $r - (4 + 6) \neq 8$

2. $26 + 3 < 19$ $5 \times (7 - 3) = 20$

3. $42 - t \neq 6$ $16 > 4 + (21 \div 3)$

4. $7 = 56 \div 8$ $m - (7 - 5) < 6$

5. $54 = 6 \times 9$ $6 = (9 \times 4) \div 6$

6. $36 > 8 \times r$ $6 = 9 \times (4 \div 6)$

7. $\frac{1}{2} > \frac{2}{5} + k$ $7 < 3 \times (b + 2)$

8. $.3 + .4 > .7$ $(28 \div 7) - 4 > 0$

9. $8 \div .5 = 4$ $28 \div (7 - 4) = 9$

10. $8 \times .5 = 4$ $(9 + c) - 5 \neq 4$

Solving Open Sentences

The solution set of an open sentence is the set of all members of the replacement set that make the open sentence become a true closed sentence.

Open sentence: $5 + a < 12$
 Replacement set: $\{5, 6, 7, 8, 9, 10\}$
 Solution set: $\{5, 6\}$

Use after page 41.

Use $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ as the replacement set. Name the solution set of each open sentence below.

- | a | b |
|-----------------------|----------------------------|
| 1. $42 \div a = 6$ | $2 \times (y + 4) = 18$ |
| 2. $t + 8 > 15$ | $(16 \div 2) - r > 1$ |
| 3. $21 - n = 9$ | $n + (7 \times 5) < 42$ |
| 4. $b < 2 \times 4$ | $a - (5 + 6) = 11$ |
| 5. $m = 81 - 62$ | $(c + 7) \times 3 = 36$ |
| 6. $72 \div d = 8$ | $(c + 7) \times 3 < 36$ |
| 7. $26 - n > 16$ | $(c + 7) \times 3 > 36$ |
| 8. $24 \div 4 < x$ | $17 - (3 \times 3) = m$ |
| 9. $r \times 7 = 84$ | $(b \div 4) \times 9 = 27$ |
| 10. $5 \times c < 21$ | $h + (9 \times 6) = 61$ |
| 11. $r + 5 \neq 13$ | $(6 + t) \times 3 \neq 36$ |

$7b$ means $7 \times b$

mn means $m \times n$

$5(7)$ means 5×7

Use after page 45.

Find the number named by each expression below if the variable is replaced as indicated.

- | | |
|-------------------------|---------------|
| 1. $(7a - 4) + 3a$ | Let $a = 5$. |
| 2. $12n - 5n$ | Let $n = 6$. |
| 3. $2c(c - 7)$ | Let $c = 8$. |
| 4. $(4r + 28) \div r$ | Let $r = 7$. |
| 5. $(5k + 7) \times 2k$ | Let $k = 4$. |
| 6. $(6t - 3t) + 2t$ | Let $t = 9$. |

Replace r by 3, s by 4, and t by 5 and tell the number named by each expression below.

- | a | b |
|------------------|----------------|
| 7. $2t - r$ | $2t + r - s$ |
| 8. $s - r + t$ | $2(r + s) - t$ |
| 9. $rs - 2t$ | $5rs \div t$ |
| 10. $3r + st$ | $2r + 3s + 4t$ |
| 11. $10r \div t$ | $(rs)t + 7$ |
| 12. $4s - 3t$ | $3r + s + 4t$ |

Addition and Subtraction

If a , b , and c represent any whole numbers, then

$$a+0=a=0+a$$

$$a+b=b+a$$

$$a+(b+c)=(a+b)+c.$$

Use after page 53.

Write *identity number*, *commutative property*, or *associative property* to indicate which is illustrated by each sentence below.

1. $(13+42)+18=13+(42+18)$

2. $2413=0+2413$

3. $261+703=703+261$

4. $(8+32)+27=27+(8+32)$

5. $(8+32)+27=8+(32+27)$

Tell which numeral should replace each variable so that the resulting sentence is true.

a

b

6. $k+0=9$ $(5+1)+c=6+(5+1)$

7. $r+5=5$ $(3+t)+7=3+(8+7)$

8. $6+0=t$ $352+87=a+352$

9. $m+7=7$ $n+(6+1)=(7+6)+1$

10. $a=0+11$ $(1+k)+8=1+8$

Let a , b , and c represent whole numbers.

If $a+b=c$, then $a=c-b$ and $b=c-a$.

Rename the minuend, if necessary, so you can subtract in every place-value position.

$$\begin{array}{r} 515 \\ 884 \\ -273 \\ \hline 381 \end{array}$$

$$\begin{array}{r} 61310 \\ 7406 \\ -2823 \\ \hline 4583 \end{array}$$

Use after page 57.

Find each difference. Check each answer by using addition.

	a	b	c
1.	$\begin{array}{r} 82 \\ -37 \\ \hline \end{array}$	$\begin{array}{r} 3172 \\ -1251 \\ \hline \end{array}$	$\begin{array}{r} 36280 \\ -14196 \\ \hline \end{array}$
2.	$\begin{array}{r} 407 \\ -352 \\ \hline \end{array}$	$\begin{array}{r} 7026 \\ -3408 \\ \hline \end{array}$	$\begin{array}{r} 28518 \\ -7426 \\ \hline \end{array}$
3.	$\begin{array}{r} 713 \\ -586 \\ \hline \end{array}$	$\begin{array}{r} 5216 \\ -888 \\ \hline \end{array}$	$\begin{array}{r} 34006 \\ -9238 \\ \hline \end{array}$
4.	$\begin{array}{r} 482 \\ -307 \\ \hline \end{array}$	$\begin{array}{r} 4008 \\ -2116 \\ \hline \end{array}$	$\begin{array}{r} 82150 \\ -50260 \\ \hline \end{array}$
5.	$\begin{array}{r} 182 \\ -92 \\ \hline \end{array}$	$\begin{array}{r} 2136 \\ -762 \\ \hline \end{array}$	$\begin{array}{r} 30502 \\ -3608 \\ \hline \end{array}$
6.	$\begin{array}{r} 710 \\ -362 \\ \hline \end{array}$	$\begin{array}{r} 8103 \\ -582 \\ \hline \end{array}$	$\begin{array}{r} 61002 \\ -7213 \\ \hline \end{array}$
7.	$\begin{array}{r} 410 \\ -208 \\ \hline \end{array}$	$\begin{array}{r} 4012 \\ -2784 \\ \hline \end{array}$	$\begin{array}{r} 50013 \\ -8764 \\ \hline \end{array}$

Multiplication

If a , b , and c represent whole numbers, then

$$\begin{aligned} ab &= ba \\ a(bc) &= (ab)c \\ a \times 1 &= a = 1 \times a. \end{aligned}$$

Use after page 61.

Write *commutative property*, *associative property*, or *identity number* to indicate which is illustrated by each sentence below.

- | a | b |
|---------------------|---|
| 1. $5 \times 1 = 5$ | $7 \times (6 \times 2) = (6 \times 2) \times 7$ |
| 2. $1 \times 7 = 7$ | $7 \times (6 \times 2) = (7 \times 6) \times 2$ |
| 3. $8 = 1 \times 8$ | $17 \times 36 = 36 \times 17$ |
| 4. $0 = 0 \times 1$ | $(1 \times 5) \times 7 = 1 \times (5 \times 7)$ |

Multiplication is distributive over addition.

$$a(b+c) = ab+ac$$

Use after page 63.

Find the simplest numeral for each product named below.

- | a | b | c |
|------------------|---------------|----------------|
| 1. 8×34 | 26×5 | 4×213 |
| 2. 7×28 | 42×7 | 8×124 |
| 3. 9×52 | 64×3 | 3×562 |
| 4. 5×47 | 82×4 | 7×326 |

The distributive property of multiplication over addition is used in the multiplication algorithm.

$$\begin{array}{r} 43 \\ \times 24 \\ \hline 172 \quad \longrightarrow \quad 4(40+3) \\ 860 \quad \longrightarrow \quad 20(40+3) \\ \hline 1032 \quad \longrightarrow \quad (20+4)(40+3) \end{array}$$

Use after page 65.

Find the simplest numeral for each product.

- | a | b | c |
|---|--|--|
| 1. $\begin{array}{r} 28 \\ \times 7 \\ \hline \end{array}$ | $\begin{array}{r} 205 \\ \times 26 \\ \hline \end{array}$ | $\begin{array}{r} 3418 \\ \times 42 \\ \hline \end{array}$ |
| 2. $\begin{array}{r} 73 \\ \times 13 \\ \hline \end{array}$ | $\begin{array}{r} 317 \\ \times 54 \\ \hline \end{array}$ | $\begin{array}{r} 5208 \\ \times 37 \\ \hline \end{array}$ |
| 3. $\begin{array}{r} 66 \\ \times 45 \\ \hline \end{array}$ | $\begin{array}{r} 570 \\ \times 762 \\ \hline \end{array}$ | $\begin{array}{r} 6139 \\ \times 308 \\ \hline \end{array}$ |
| 4. $\begin{array}{r} 93 \\ \times 36 \\ \hline \end{array}$ | $\begin{array}{r} 487 \\ \times 215 \\ \hline \end{array}$ | $\begin{array}{r} 4027 \\ \times 3131 \\ \hline \end{array}$ |
| 5. $\begin{array}{r} 57 \\ \times 30 \\ \hline \end{array}$ | $\begin{array}{r} 816 \\ \times 270 \\ \hline \end{array}$ | $\begin{array}{r} 2826 \\ \times 3005 \\ \hline \end{array}$ |
| 6. $\begin{array}{r} 47 \\ \times 29 \\ \hline \end{array}$ | $\begin{array}{r} 903 \\ \times 435 \\ \hline \end{array}$ | $\begin{array}{r} 7030 \\ \times 515 \\ \hline \end{array}$ |
| 7. $\begin{array}{r} 68 \\ \times 44 \\ \hline \end{array}$ | $\begin{array}{r} 807 \\ \times 206 \\ \hline \end{array}$ | $\begin{array}{r} 4521 \\ \times 660 \\ \hline \end{array}$ |
| 8. $\begin{array}{r} 76 \\ \times 18 \\ \hline \end{array}$ | $\begin{array}{r} 343 \\ \times 49 \\ \hline \end{array}$ | $\begin{array}{r} 8080 \\ \times 3535 \\ \hline \end{array}$ |

Division

Let a , b , and c represent whole numbers.

If $ab=c$, then $a=\frac{c}{b}$ and $b=\frac{c}{a}$.

$0 \div a = 0$ if $a \neq 0$.

Division by 0 is meaningless.

If $a \neq 0$ and a is a factor of b , then b is divisible by a .

$$(a+b) \div c = (a \div c) + (b \div c)$$

Use after page 71.

Solve each open sentence below.

- | a | b |
|-----------------------|-----------------------------|
| 1. $9a=54$ | $(15 \div x) \times 7 = 21$ |
| 2. $r \div 7 = 15$ | $32 = (t+4) \div 4$ |
| 3. $n \times 14 = 98$ | $39 = (18 \div 6) \times k$ |
| 4. $84 \div a = 7$ | $b = (21 \div 7) \times 7$ |
| 5. $15c = 0$ | $a = (22 - 22) \div 9$ |

Distribute division over addition to find the simplest numeral for each quotient below.

- | a | b | c |
|----------------|-------------|-------------|
| 6. $48 \div 4$ | $84 \div 7$ | $72 \div 6$ |
| 7. $94 \div 4$ | $24 \div 2$ | $52 \div 4$ |
| 8. $85 \div 5$ | $98 \div 7$ | $88 \div 8$ |
| 9. $90 \div 5$ | $78 \div 6$ | $75 \div 3$ |

By inspecting a decimal numeral you can determine whether the number is divisible by 2, 3, 5, or 9.

Use after page 80.

Copy. If a numeral names a number that is divisible by

- 2, underline it,
- 3, draw a ring around it,
- 5, write \checkmark above it,
- 9, draw an x through it.

- | | a | b | c | d |
|----|-----|------|------|-------|
| 1. | 682 | 7236 | 3105 | 71820 |
| 2. | 405 | 3471 | 5003 | 30526 |
| 3. | 160 | 2345 | 6257 | 11022 |
| 4. | 360 | 9876 | 3210 | 50247 |

Every composite number can be expressed as a product of primes.

$$84 = 4 \times 21 = 2 \times 2 \times 3 \times 7 = 2^2 \times 3 \times 7$$

Use after page 83.

Express each number as a product of primes.

- | | a | b | c | d |
|----|-----|-----|-----|-----|
| 1. | 56 | 54 | 210 | 126 |
| 2. | 48 | 80 | 294 | 525 |
| 3. | 105 | 168 | 375 | 900 |

Integers

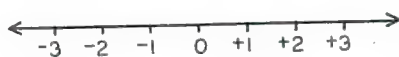
$+3$ means 3 units in the positive direction and -5 means 5 units in the negative direction.

Use after page 91.

Write a numeral to describe each of these moves on a number line.

- | a | b |
|----------------------|-------------------|
| 1. from $+1$ to $+8$ | from $+2$ to 0 |
| 2. from -1 to -8 | from -5 to $+5$ |
| 3. from 0 to $+7$ | from $+1$ to -4 |
| 4. from $+6$ to $+5$ | from -1 to $+4$ |

The greater of two integers is represented farther to the right on a horizontal number line.



$$-3 < +2 \quad \text{or} \quad +2 > -3$$

Use after page 93.

Replace each \square with $<$, $=$, or $>$ so the resulting sentence is true.

- | a | b | c |
|--------------------|-----------------|-------------------|
| 1. $3 \square 8$ | $-4 \square -4$ | $17 \square 21$ |
| 2. $-3 \square 8$ | $-4 \square 4$ | $-15 \square -13$ |
| 3. $-3 \square -8$ | $4 \square -4$ | $-19 \square 8$ |
| 4. $3 \square -8$ | $4 \square 4$ | $0 \square -7$ |

If a , b , and c represent integers, then

$$\begin{aligned} a+b &= b+a \\ a+(b+c) &= (a+b)+c \\ a+(-a) &= 0 = -a+a. \end{aligned}$$

Use after page 97.

Solve each open sentence below.

- | a | b |
|-------------------|-----------------|
| 1. $+5 + +6 = a$ | $+5 + -6 = b$ |
| 2. $-3 + -4 = c$ | $-3 + +4 = d$ |
| 3. $-14 + +6 = e$ | $+14 + -6 = q$ |
| 4. $-9 + -16 = n$ | $+12 + -12 = r$ |
| 5. $+8 + -7 = s$ | $-12 + +5 = t$ |

Solve each open sentence below.

6. $-3 + (+5 + -4) = y$
7. $(+7 + +6) + -9 = x$
8. $+6 + (-13 + +7) = n$
9. $+9 + -6 = r + +9$
10. $-17 + s = 0$
11. $(-7 + -3) + -8 = t$
12. $(+4 + -3) + +3 = a$
13. $c = -7 + (+5 + +2)$
14. $-8 + m = +13 + -8$

Integers

Subtracting an integer can be done by adding its opposite.

$$+5 - +3 = +5 + -3 = +2$$

$$-7 - -8 = -7 + +8 = +1$$

$$+2 - +6 = +2 + -6 = -4$$

Use after page 101.

Write an open addition sentence for each subtraction sentence below. Then solve each open addition sentence.

a

b

1. $+12 - +6 = a$ $n = +7 - -5$

2. $-9 - -9 = r$ $t = -31 - +7$

3. $+15 - -12 = s$ $b = -4 - -17$

4. $+5 - +24 = d$ $c = -17 - 0$

5. $-13 - +9 = e$ $m = +3 - +23$

6. $-7 - -5 = x$ $y = 0 - +11$

7. $+31 - +31 = k$ $n = -27 - -26$

8. $+9 - a = +2$ $-5 - c = +6$

9. $d - +2 = 0$ $+10 = -17 - r$

10. $-12 - s = +5$ $b = +8 - -18$

11. $+30 - -20 = n$ $m - 0 = +12$

12. $-30 - +20 = k$ $x - 0 = -12$

The product of two integers is

a. positive if both factors are positive or both factors are negative,

b. negative if one factor is positive and the other is negative.

The quotient of two integers is

a. positive if the dividend and the divisor are both positive or both negative,

b. negative if either the dividend or the divisor is positive and the other is negative.

Use after page 104.

Find the simplest numeral for each expression below.

a

b

c

1. $-7 \times +9$ $+7 \times -9$ -7×-9

2. $+12 \times +4$ -12×-4 $-12 \times +4$

3. $-3 \times +11$ $+3 \times -11$ -3×-11

4. $+14 \times +2$ $-14 \times +2$ $+14 \times -2$

5. -15×0 0×-15 $+15 \times 0$

6. $+6 \times -12$ -6×-12 $+6 \times +12$

7. $-24 \div -3$ $-24 \div +3$ $+24 \div +3$

8. $+32 \div +8$ $+32 \div -8$ $-32 \div +8$

9. $-54 \div +6$ $-54 \div -6$ $+54 \div -6$

Multiplication of Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers, then

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Use after page 113.

Find a single fraction for each expression below.

- | | a | b | c |
|----|-----------------------------------|-------------------------|------------------------------------|
| 1. | $\frac{2}{7} \times \frac{5}{3}$ | $5 \times \frac{3}{4}$ | $\frac{2}{7} \times \frac{4}{9}$ |
| 2. | $\frac{5}{8} \times \frac{7}{9}$ | $\frac{5}{6} \times 7$ | $\frac{11}{12} \times \frac{3}{4}$ |
| 3. | $\frac{13}{7} \times \frac{4}{5}$ | $\frac{7}{8} \times 5$ | $\frac{3}{11} \times \frac{1}{4}$ |
| 4. | $\frac{1}{3} \times \frac{1}{14}$ | $6 \times \frac{2}{17}$ | $\frac{4}{7} \times \frac{6}{5}$ |
| 5. | $\frac{8}{9} \times \frac{2}{3}$ | $11 \times \frac{3}{5}$ | $\frac{7}{10} \times \frac{9}{3}$ |

The greatest common factor of two or more numbers is the greatest number that is a factor of each of the numbers.

$$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$$

$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$

The greatest common factor of 24 and 36 is $2^2 \times 3 = 4 \times 3$ or 12.

Use after page 117.

Find the greatest common factor of the numbers in each set below.

- | | a | b | c |
|----|----------|----------|--------------|
| 1. | {70, 42} | {30, 18} | {12, 24, 32} |
| 2. | {63, 84} | {30, 75} | {24, 18, 45} |

3. {27, 54} {27, 30} {10, 24, 32}

4. {16, 28} {36, 54} {48, 60, 36}

5. {24, 32} {30, 42} {36, 54, 90}

A fraction is in simplest form if the numerator and denominator are relatively prime. A product can be found in simplest form as shown below.

$$\frac{5}{\underset{1}{\cancel{6}}} \times \frac{\overset{2}{\cancel{12}}}{17} = \frac{10}{17}$$

Use after page 121.

Change each product below to simplest form.

- | | a | b |
|-----|--------------------------------------|--|
| 1. | $\frac{5}{6} \times \frac{2}{5}$ | $\frac{4}{9} \times \frac{3}{8}$ |
| 2. | $\frac{6}{7} \times \frac{7}{6}$ | $\frac{3}{4} \times \frac{5}{6}$ |
| 3. | $\frac{3}{5} \times \frac{10}{21}$ | $\frac{6}{3} \times \frac{9}{8}$ |
| 4. | $\frac{7}{10} \times \frac{5}{14}$ | $\frac{16}{21} \times \frac{7}{8}$ |
| 5. | $\frac{4}{27} \times \frac{18}{31}$ | $\frac{5}{8} \times \frac{3}{10} \times \frac{2}{7}$ |
| 6. | $\frac{11}{18} \times \frac{32}{33}$ | $\frac{3}{4} \times \frac{8}{9} \times \frac{7}{10}$ |
| 7. | $\frac{20}{27} \times \frac{18}{5}$ | $\frac{3}{8} \times \frac{12}{15} \times \frac{5}{16}$ |
| 8. | $\frac{7}{3} \times \frac{27}{35}$ | $\frac{5}{6} \times \frac{24}{35} \times \frac{3}{8}$ |
| 9. | $\frac{4}{21} \times \frac{15}{16}$ | $\frac{6}{7} \times \frac{9}{10} \times \frac{7}{12}$ |
| 10. | $\frac{9}{2} \times \frac{36}{63}$ | $\frac{1}{2} \times \frac{3}{5} \times \frac{7}{11}$ |

Rational Numbers

A mixed numeral expresses a sum.

$$7\frac{1}{2} = 7 + \frac{1}{2}$$

When a factor is named by a mixed numeral, you have several choices as to how to compute the product.

$$5\frac{1}{2} \times \frac{3}{4} = \frac{11}{2} \times \frac{3}{4} = \frac{33}{8} \text{ or } 4\frac{1}{8}$$

$$3 \times 4\frac{1}{5} = 3(4 + \frac{1}{5}) = 12 + \frac{3}{5} = 12\frac{3}{5}$$

Use after page 125.

Express each product below in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $7 \times 3\frac{1}{2}$	$5\frac{2}{3} \times \frac{3}{4}$	$2\frac{1}{2} \times 3\frac{2}{3}$
2. $5\frac{3}{5} \times 15$	$\frac{5}{6} \times 2\frac{2}{5}$	$7\frac{3}{4} \times 5\frac{5}{6}$
3. $4\frac{3}{10} \times 16$	$11\frac{1}{3} \times \frac{9}{17}$	$7\frac{1}{4} \times 6\frac{3}{7}$
4. $8 \times 9\frac{1}{2}$	$1\frac{4}{5} \times \frac{5}{6}$	$9\frac{5}{7} \times 2\frac{3}{4}$
5. $5\frac{5}{6} \times 4$	$12\frac{2}{3} \times \frac{1}{2}$	$5\frac{7}{10} \times 1\frac{2}{3}$
6. $9 \times 4\frac{2}{3}$	$3\frac{4}{5} \times \frac{6}{7}$	$3\frac{1}{2} \times 2\frac{1}{4}$
7. $7\frac{3}{8} \times \frac{5}{16}$	$1\frac{1}{5} \times 15$	$5\frac{5}{8} \times 6\frac{2}{5}$
8. $4\frac{1}{2} \times \frac{1}{4}$	$19\frac{1}{2} \times 16$	$2\frac{4}{5} \times 1\frac{1}{4}$
9. $3\frac{9}{10} \times 20$	$12 \times 70\frac{5}{8}$	$7\frac{3}{8} \times 4\frac{5}{16}$
10. $2\frac{1}{3} \times 6\frac{1}{8}$	$5\frac{5}{8} \times 4\frac{3}{4}$	$20\frac{2}{3} \times 10\frac{1}{2}$
11. $5\frac{1}{6} \times \frac{3}{4}$	$8\frac{2}{7} \times 2\frac{2}{3}$	$15\frac{3}{5} \times 1\frac{4}{13}$
12. $3\frac{1}{2} \times 3\frac{1}{2}$	$\frac{7}{8} \times 6\frac{3}{4}$	$7\frac{1}{4} \times 7\frac{1}{4}$

If $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Use after page 129.

Express each quotient below in simplest form.

<i>a</i>	<i>b</i>	<i>c</i>
1. $\frac{7}{8} \div \frac{3}{8}$	$\frac{3}{5} \div \frac{4}{7}$	$\frac{5}{9} \div \frac{1}{4}$
2. $\frac{3}{7} \div \frac{1}{5}$	$2\frac{1}{3} \div \frac{1}{2}$	$\frac{1}{3} \div 2\frac{1}{6}$
3. $\frac{5}{8} \div \frac{4}{7}$	$8 \div \frac{6}{7}$	$1\frac{3}{4} \div \frac{7}{10}$
4. $\frac{6}{5} \div \frac{9}{10}$	$1\frac{1}{5} \div \frac{2}{3}$	$3\frac{1}{2} \div 3\frac{1}{10}$
5. $0 \div 3\frac{1}{2}$	$12 \div \frac{4}{9}$	$4\frac{1}{4} \div 2\frac{1}{3}$

Solve each problem below.

6. A druggist used $7\frac{1}{2}$ ounces of medicine to fill 5 bottles. If each bottle contains the same amount, how many ounces of medicine are there in each bottle?

7. A boat traveled $17\frac{1}{2}$ miles in 1 hour. At that speed, how far will the boat travel in $5\frac{1}{4}$ hours?

8. A box contained 6 hand drills with a combined weight of $31\frac{1}{2}$ pounds. What was the weight of each of these hand drills?

Rational Numbers

Let $\frac{a}{b}$ and $\frac{c}{d}$ represent rational numbers.

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

If $\frac{a}{b} < \frac{c}{d}$, then $ad < bc$.

If $\frac{a}{b} > \frac{c}{d}$, then $ad > bc$.

Use after page 137.

Replace each \square with $=$ or \neq so the resulting sentence is true.

- | | a | b | c |
|----|---------------------------------------|--------------------------------------|---------------------------------------|
| 1. | $\frac{2}{3} \square \frac{3}{4}$ | $\frac{7}{8} \square \frac{3}{4}$ | $\frac{5}{15} \square \frac{9}{27}$ |
| 2. | $\frac{12}{15} \square \frac{21}{25}$ | $\frac{4}{10} \square \frac{14}{35}$ | $\frac{10}{14} \square \frac{15}{20}$ |
| 3. | $\frac{2}{5} \square \frac{21}{50}$ | $\frac{5}{7} \square \frac{35}{49}$ | $\frac{11}{16} \square \frac{5}{8}$ |

Replace each \square with $<$, $=$, or $>$ so the resulting sentence is true.

- | | a | b | c |
|----|-------------------------------------|--------------------------------------|---------------------------------------|
| 4. | $\frac{5}{8} \square \frac{3}{4}$ | $\frac{4}{13} \square \frac{9}{26}$ | $\frac{2}{3} \square \frac{11}{15}$ |
| 5. | $\frac{11}{15} \square \frac{2}{3}$ | $\frac{8}{9} \square \frac{7}{8}$ | $\frac{11}{10} \square \frac{16}{15}$ |
| 6. | $\frac{4}{7} \square \frac{5}{9}$ | $\frac{7}{10} \square \frac{21}{30}$ | $\frac{4}{13} \square \frac{5}{16}$ |

Solve each open sentence below.

- | | a | b | c |
|----|--------------------------------|-------------------------------|--------------------------------|
| 7. | $\frac{7}{10} = \frac{n}{30}$ | $\frac{14}{n} = \frac{7}{5}$ | $\frac{n}{54} = \frac{40}{27}$ |
| 8. | $\frac{5}{11} = \frac{n}{66}$ | $\frac{4}{n} = \frac{12}{45}$ | $\frac{36}{8} = \frac{27}{n}$ |
| 9. | $\frac{14}{30} = \frac{35}{n}$ | $\frac{n}{21} = \frac{15}{9}$ | $\frac{17}{n} = \frac{51}{39}$ |

To add or subtract rational numbers with different denominators, rename them so they have a common denominator.

$$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{6+5}{15} = \frac{11}{15}$$

$$\begin{array}{r} 3\frac{5}{6} \\ + 5\frac{1}{8} \\ \hline \end{array} \rightarrow \begin{array}{r} 3\frac{20}{24} \\ + 5\frac{3}{24} \\ \hline 8\frac{23}{24} \end{array}$$

$$\frac{7}{8} - \frac{1}{2} = \frac{7}{8} - \frac{4}{8} = \frac{7-4}{8} = \frac{3}{8}$$

$$\begin{array}{r} 7\frac{3}{4} \\ - 3\frac{2}{5} \\ \hline \end{array} \rightarrow \begin{array}{r} 7\frac{15}{20} \\ - 3\frac{8}{20} \\ \hline 4\frac{7}{20} \end{array}$$

Use after page 149.

Express each sum or difference in simplest form.

- | | a | b | c |
|----|---|--|--|
| 1. | $\frac{3}{2} + \frac{5}{6}$ | $\frac{4}{5} - \frac{3}{8}$ | $\frac{6}{7} + \frac{5}{6}$ |
| 2. | $\frac{4}{9} + \frac{1}{6}$ | $\frac{4}{6} - \frac{5}{8}$ | $\frac{3}{4} + \frac{7}{8}$ |
| 3. | $\frac{1}{6} + \frac{1}{7}$ | $\frac{7}{10} - \frac{1}{4}$ | $\frac{1}{2} - \frac{5}{12}$ |
| 4. | $\frac{5}{8} + \frac{4}{9}$ | $\frac{5}{4} + \frac{7}{12}$ | $\frac{6}{15} - \frac{1}{10}$ |
| 5. | $9\frac{7}{12} + 6\frac{5}{6}$ | $7\frac{3}{8} - 6\frac{1}{4}$ | $58\frac{2}{3} - 29\frac{4}{5}$ |
| 6. | $5\frac{4}{5} + 3\frac{7}{10}$ | $18\frac{1}{4} - 9\frac{2}{3}$ | $13\frac{4}{9} - 6\frac{5}{6}$ |
| 7. | $2\frac{1}{6} + 4\frac{3}{18} + 9\frac{2}{3}$ | $26\frac{3}{4} + 7\frac{1}{6} + 11\frac{2}{5}$ | $32\frac{1}{8} + 9\frac{2}{3} + 15\frac{5}{6}$ |

Decimals

In addition or subtraction with decimals, arrange the decimals so that the decimal points align vertically. Then add or subtract as with whole numbers.

$$\begin{array}{r} 36.014 \\ 7.28 \\ +14.9 \\ \hline 58.194 \end{array} \qquad \begin{array}{r} 62.7 \\ -29.38 \\ \hline 33.32 \end{array}$$

Use after page 167.

Copy. Find the simplest numeral for each sum or difference.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\begin{array}{r} 54.8 \\ +36.4 \\ \hline \end{array}$	$\begin{array}{r} 7.26 \\ +3.972 \\ \hline \end{array}$	$\begin{array}{r} 43.028 \\ +8.47 \\ \hline \end{array}$
2.	$\begin{array}{r} 37.3 \\ 29.6 \\ +18.7 \\ \hline \end{array}$	$\begin{array}{r} 5.7 \\ .576 \\ +2.08 \\ \hline \end{array}$	$\begin{array}{r} 29.306 \\ 7.8 \\ +8.74 \\ \hline \end{array}$
3.	$\begin{array}{r} 92.7 \\ 3.5 \\ +21.3 \\ \hline \end{array}$	$\begin{array}{r} 8.25 \\ .307 \\ +1.826 \\ \hline \end{array}$	$\begin{array}{r} 42.18 \\ 36.092 \\ +51.439 \\ \hline \end{array}$
4.	$\begin{array}{r} 36.7 \\ -28.5 \\ \hline \end{array}$	$\begin{array}{r} 9.21 \\ -5.036 \\ \hline \end{array}$	$\begin{array}{r} 74.05 \\ -6.241 \\ \hline \end{array}$
5.	$\begin{array}{r} 24.3 \\ -18.6 \\ \hline \end{array}$	$\begin{array}{r} 6.204 \\ -3.65 \\ \hline \end{array}$	$\begin{array}{r} 82.307 \\ -49.519 \\ \hline \end{array}$
6.	$\begin{array}{r} 70.1 \\ -58.5 \\ \hline \end{array}$	$\begin{array}{r} 5.321 \\ -4.607 \\ \hline \end{array}$	$\begin{array}{r} 70.301 \\ -9.67 \\ \hline \end{array}$
7.	$\begin{array}{r} 32.1 \\ -20.8 \\ \hline \end{array}$	$\begin{array}{r} 8.003 \\ -5.296 \\ \hline \end{array}$	$\begin{array}{r} 20.005 \\ -8.207 \\ \hline \end{array}$

In multiplication with decimals, multiply as with whole numbers and place the decimal point in the product numeral according to the number of decimal places in both factor numerals.

$$\begin{array}{r} 41.7 \rightarrow 1 \text{ digit} \\ \times .23 \rightarrow 2 \text{ digits} \\ \hline 1251 \\ 834 \\ \hline 9.591 \leftarrow 3 \text{ digits} \end{array}$$

Use after page 171.

Copy. Find the simplest numeral for each product.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\begin{array}{r} 38.3 \\ \times 2.7 \\ \hline \end{array}$	$\begin{array}{r} 5.36 \\ \times 5.5 \\ \hline \end{array}$	$\begin{array}{r} 41.24 \\ \times 3.6 \\ \hline \end{array}$
2.	$\begin{array}{r} 5.42 \\ \times .15 \\ \hline \end{array}$	$\begin{array}{r} 72.1 \\ \times .43 \\ \hline \end{array}$	$\begin{array}{r} 50.34 \\ \times .08 \\ \hline \end{array}$
3.	$\begin{array}{r} .417 \\ \times 5.8 \\ \hline \end{array}$	$\begin{array}{r} 742 \\ \times .32 \\ \hline \end{array}$	$\begin{array}{r} 432.7 \\ \times 8.4 \\ \hline \end{array}$
4.	$\begin{array}{r} 300 \\ \times .09 \\ \hline \end{array}$	$\begin{array}{r} .015 \\ \times 4.2 \\ \hline \end{array}$	$\begin{array}{r} 6.132 \\ \times 8.5 \\ \hline \end{array}$
5.	$\begin{array}{r} .048 \\ \times .25 \\ \hline \end{array}$	$\begin{array}{r} 7.35 \\ \times 2.4 \\ \hline \end{array}$	$\begin{array}{r} 314.7 \\ \times .38 \\ \hline \end{array}$
6.	$\begin{array}{r} .142 \\ \times 40 \\ \hline \end{array}$	$\begin{array}{r} 17.9 \\ \times .47 \\ \hline \end{array}$	$\begin{array}{r} 19.34 \\ \times .46 \\ \hline \end{array}$
7.	$\begin{array}{r} 28.4 \\ \times 3.5 \\ \hline \end{array}$	$\begin{array}{r} 4.56 \\ \times .55 \\ \hline \end{array}$	$\begin{array}{r} 37.45 \\ \times 1.8 \\ \hline \end{array}$

Decimals, Ratio, Proportion

In division with decimals, express the division so the divisor is a whole number.

$$\begin{array}{r} .5 \overline{) 1.25} \\ 5 \overline{) 12.5} \\ \underline{10} \\ 25 \\ \underline{25} \\ 0 \end{array}$$

Use after page 177.

Copy. Find the simplest numeral for each quotient.

a	b
1. $.8 \overline{) 2.88}$	$.09 \overline{) 3.483}$
2. $.05 \overline{) .170}$	$.16 \overline{) 19.872}$
3. $2.3 \overline{) 15.18}$	$.024 \overline{) .29424}$
4. $.32 \overline{) .464}$	$.012 \overline{) 2.976}$
5. $3.4 \overline{) 295.8}$	$1.18 \overline{) 3.8114}$
6. $.6 \overline{) 54.78}$	$.021 \overline{) .6405}$
7. $.14 \overline{) 4.578}$	$.28 \overline{) .00336}$
8. $.23 \overline{) .2576}$	$3.9 \overline{) 417.3}$
9. $.07 \overline{) 1.022}$	$2.15 \overline{) .5805}$
10. $.13 \overline{) .0364}$	$17.8 \overline{) .712}$
11. $.08 \overline{) 10.08}$	$.015 \overline{) .0585}$

A proportion can be solved by considering the ratios as equivalent rational numbers.

$$\begin{aligned} \frac{5}{8} &= \frac{65}{x} \\ 5x &= 65 \times 8 \\ x &= \frac{65 \times 8}{5} \\ x &= 104 \end{aligned}$$

Use after page 195.

Copy. Solve each of the following proportions.

a	b	c
1. $\frac{4}{9} = \frac{n}{45}$	$\frac{3}{5} = \frac{27}{r}$	$\frac{a}{15} = \frac{6}{80}$
2. $\frac{5}{n} = \frac{13}{78}$	$\frac{7}{12} = \frac{a}{48}$	$\frac{7}{8} = \frac{126}{c}$
3. $\frac{a}{42} = \frac{50}{75}$	$\frac{16}{r} = \frac{28}{49}$	$\frac{27}{72} = \frac{r}{104}$
4. $\frac{15}{48} = \frac{35}{n}$	$\frac{a}{28} = \frac{24}{32}$	$\frac{30}{12} = \frac{45}{s}$
5. $\frac{24}{30} = \frac{n}{45}$	$\frac{14}{a} = \frac{21}{48}$	$\frac{r}{36} = \frac{55}{60}$

Solve each problem below.

6. A rear wheel on a tricycle makes 12 revolutions while the front wheel makes 7 revolutions. How many revolutions does a rear wheel make while the front wheel makes 49 revolutions?

7. John had 12 hits in his last 40 times at bat. At this rate, how many hits can he expect to make in his next 30 times at bat?

Per Cent

A rational number can be expressed as a fraction, a decimal, or a per cent.

$$\frac{2}{5} = \frac{2}{5} \times \frac{2}{2} = \frac{4}{10} = .4$$

$$.4 = \frac{4}{10} \times \frac{10}{10} = \frac{40}{100} \text{ or } 40\%$$

Use after page 199.

Copy. Complete the table so that all numerals in each row name the same number. Express all fractions in simplest form.

	<i>Fraction</i>	<i>Decimal</i>	<i>Per Cent</i>
1.	$\frac{4}{5}$	_____	_____
2.	_____	_____	28%
3.	_____	.76	_____
4.	$\frac{7}{50}$	_____	_____
5.	_____	2.14	_____
6.	_____	_____	$\frac{1}{2}\%$
7.	_____	$.005\frac{1}{2}$	_____
8.	$1\frac{3}{4}$	_____	_____
9.	_____	_____	324%
10.	$12\frac{5}{8}$	_____	_____
11.	_____	$.6\frac{3}{8}$	_____
12.	_____	_____	.006%

A per cent problem can be solved by using a proportion.

8 is 15% of what number?

$$\frac{8}{n} = \frac{15}{100}$$

4 is what per cent of 32?

$$\frac{4}{32} = \frac{n}{100}$$

What number is 14% of 18?

$$\frac{n}{18} = \frac{14}{100}$$

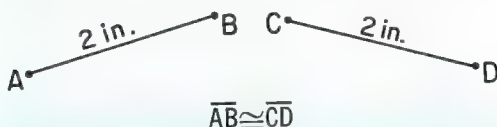
Use after page 203.

Solve each problem below.

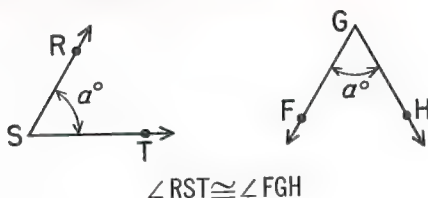
- What number is 75% of 18?
- 9 is what per cent of 60?
- 30 is 120% of what number?
- There were 15 problems on a test. Sally answered 80% of them correctly. How many problems did she answer correctly?
- A camera priced to sell for \$18.50 is on sale at a 10% discount. Disregarding tax, what would you pay for the camera during the sale?
- In a class of 30 pupils, 12 pupils are on the honor roll. What per cent of the pupils are on the honor roll?
- In a school of 520 pupils, only $2\frac{1}{2}\%$ had perfect attendance. How many pupils had perfect attendance in that school?

Lines, Circles, Angles

Two line segments are congruent if they have the same length.



Two angles are congruent if they have the same measure.



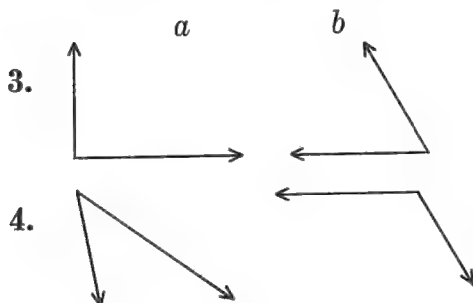
Use after page 219.

For each line segment below, construct a line segment that is congruent to it.

1. _____

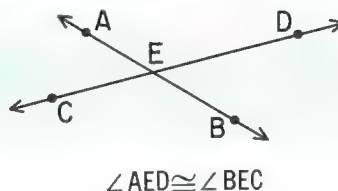
2. _____

Draw angles like those below. For each angle you draw, construct an angle that is congruent to it.



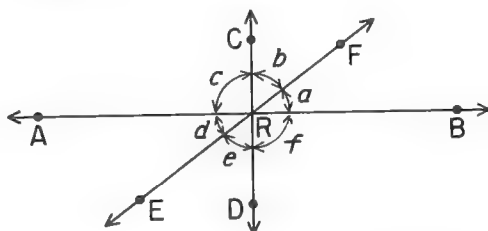
Angles are classified as acute, right, obtuse, and straight angles.

If two lines intersect, the vertical angles are congruent.



Use after page 226.

In the figure below, $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$.



Complete each sentence below so that it becomes true.

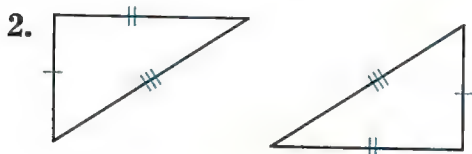
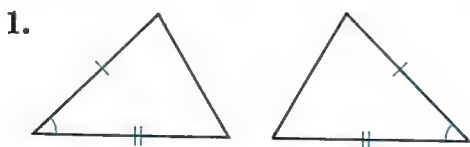
- $m\angle BRC = \underline{\hspace{2cm}}$
- $\angle a$ is an $\underline{\hspace{2cm}}$ angle.
- $\angle FRD$ is an $\underline{\hspace{2cm}}$ angle.
- $\angle b$ and $\angle \underline{\hspace{2cm}}$ are vertical angles.
- If $m\angle d = 47$, then $m\angle a = \underline{\hspace{2cm}}$.
- $m\angle b + m\angle a = \underline{\hspace{2cm}}$
- $m\angle d + m\angle e + m\angle f = \underline{\hspace{2cm}}$

Polygons

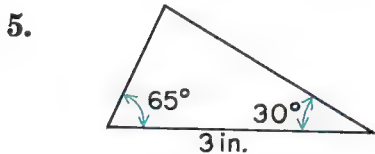
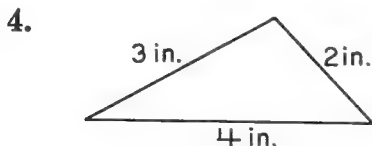
Two triangles are congruent if their corresponding sides and corresponding angles are congruent.

Use after page 237.

Write a statement that tells why the triangles are congruent in each exercise below.



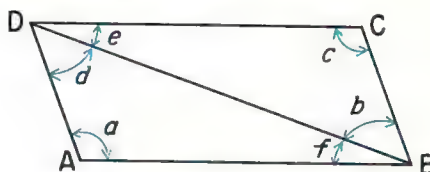
Use a ruler and a protractor to draw a triangle like each of those below. Then construct a triangle congruent to each triangle you drew.



Opposite sides of a parallelogram are parallel and congruent. Opposite angles of a parallelogram are congruent.

Use after page 246.

Complete each sentence below about parallelogram ABCD.



1. If $m\angle a = 123$, then $m\angle c = \underline{\hspace{1cm}}$.
2. If \overline{AD} is 7 in. long, then \overline{BC} is $\underline{\hspace{1cm}}$ in. long.
3. If $m\angle e = 28$, then $m\angle f = \underline{\hspace{1cm}}$.
4. If \overline{DC} is 11 cm. long, then \overline{AB} is $\underline{\hspace{1cm}}$ cm. long.
5. If $m\angle e + m\angle d = 70$, then $m\angle b + m\angle f = \underline{\hspace{1cm}}$.

Write a reason for each statement below about parallelogram ABCD.

6. $\angle a \cong \angle c$
7. $\overline{AD} \cong \overline{BC}$
8. $\overline{DC} \cong \overline{AB}$
9. $\triangle ABD \cong \triangle CDB$

Units of Measurement

A table of measures is useful for changing a measurement to another number of different units.

$$\begin{array}{ll} 12 \text{ in.} = 1 \text{ ft.} & 36 \text{ in.} = 1 \text{ yd.} \\ 3 \text{ ft.} = 1 \text{ yd.} & 5280 \text{ ft.} = 1 \text{ mi.} \end{array}$$

Use after page 249.

Complete each sentence so that it becomes true.

- | a | b |
|---------------------------------|------------------------------|
| 1. 7 ft. = ___ in. | 132 in. = ___ ft. |
| 2. $3\frac{1}{2}$ ft. = ___ in. | 72 ft. = ___ yd. |
| 3. 4 mi. = ___ ft. | $5\frac{1}{3}$ yd. = ___ ft. |
| 4. 65 ft. = ___ yd. | $7\frac{1}{4}$ ft. = ___ in. |
| 5. 5 yd. = ___ in. | 144 in. = ___ yd. |

The metric system is patterned after the decimal numeration system.

$$\begin{array}{ll} 1 \text{ m.} = 10 \text{ dm.} & 10 \text{ m.} = 1 \text{ dk.} \\ 1 \text{ dm.} = 10 \text{ cm.} & 100 \text{ m.} = 1 \text{ hm.} \\ 1 \text{ cm.} = 10 \text{ mm.} & 1000 \text{ m.} = 1 \text{ km.} \end{array}$$

Use after page 251.

Complete each sentence so that it becomes true.

- | a | b |
|-------------------|-------------------|
| 1. 4 m. = ___ cm. | 650 mm. = ___ cm. |
| 2. 7 km. = ___ m. | 250 cm. = ___ dm. |

$$3. \quad 7 \text{ cm.} = \text{___ mm.} \quad 925 \text{ cm.} = \text{___ m.}$$

$$4. \quad 3.1 \text{ m.} = \text{___ cm.} \quad 52.7 \text{ m.} = \text{___ km.}$$

$$5. \quad 18 \text{ mm.} = \text{___ cm.} \quad 8.6 \text{ dm.} = \text{___ mm.}$$

The formulas below are used to change a Fahrenheit reading to a centigrade reading and vice versa.

$$C = \frac{5}{9}(F - 32) \quad F = \frac{9}{5}C + 32$$

Use after page 253.

Use the formulas above to change each temperature reading below as indicated.

<i>Fahrenheit reading</i>	<i>Centigrade reading</i>
1. 77°F.	_____
2. _____	65°C.
3. _____	215°C.
4. 149°F.	_____
5. 212°F.	_____
6. _____	310°C.
7. 185°F.	_____
8. 320°F.	_____
9. _____	95°C.
10. _____	165°C.

Measurement

A measurement can be made only to the nearest unit. The range of the measurement is from $\frac{1}{2}$ unit less to $\frac{1}{2}$ unit more than the measurement.

Measurement: $2\frac{1}{4}$ ft.

Range of measurement: $2\frac{1}{8}$ ft. to $2\frac{3}{8}$ ft.

Use after page 259.

Record the unit of measure and the range of measurement for each measurement below.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	38 ft.	3.8 ft.	.38 ft.
2.	7.2 in.	$3\frac{1}{4}$ in.	5.0 in.
3.	.23 mi.	$2\frac{1}{2}$ mi.	217 mi.
4.	$3\frac{1}{2}$ hr.	$3\frac{2}{4}$ hr.	3.5 hr.
5.	17.2 m.	1.72 m.	172 m.

Which measurement in each pair of measurements below has the greater precision?

	<i>a</i>	<i>b</i>
6.	$2\frac{1}{2}$ in., $3\frac{1}{4}$ in.	7.1 mi., 9 mi.
7.	4 ft., $5\frac{0}{2}$ ft.	2 in., 3.8 in.
8.	$1\frac{3}{16}$ in., $1\frac{3}{8}$ in.	.14 m., 140 m.
9.	$3\frac{0}{4}$ ft., 3.0 ft.	$41\frac{1}{2}$ yd., 111 yd.

The sum or the difference of two measures is recorded with the same precision as that of the least precise measurement.

$$\begin{array}{r} 28.6 \text{ in.} \\ +13 \text{ in.} \\ \hline \end{array}$$

$$\begin{array}{r} 29 \text{ in.} \\ +13 \text{ in.} \\ \hline 42 \text{ in.} \end{array}$$

Use after page 263.

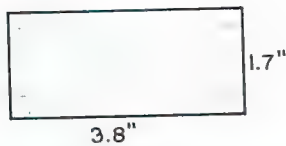
Compute each sum or difference.

	<i>a</i>	<i>b</i>
1.	$\begin{array}{r} 34.2 \text{ in.} \\ +16 \text{ in.} \\ \hline \end{array}$	$\begin{array}{r} 29.3 \text{ mi.} \\ +36.18 \text{ mi.} \\ \hline \end{array}$
2.	$\begin{array}{r} 16.0 \text{ ft.} \\ -47.8 \text{ ft.} \\ \hline \end{array}$	$\begin{array}{r} 123 \text{ yd.} \\ -89.5 \text{ yd.} \\ \hline \end{array}$
3.	$\begin{array}{r} 85.05 \text{ ft.} \\ 38.764 \text{ ft.} \\ +13.9 \text{ ft.} \\ \hline \end{array}$	$\begin{array}{r} 74.36 \text{ mi.} \\ 3.0 \text{ mi.} \\ +26.9 \text{ mi.} \\ \hline \end{array}$

Compute each sum or difference.

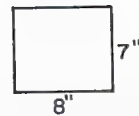
4.	$\begin{array}{r} 18 \text{ hr. } 37 \text{ min. } 46 \text{ sec.} \\ + (9 \text{ hr. } 24 \text{ min. } 19 \text{ sec.}) \\ \hline \end{array}$
5.	$\begin{array}{r} 4 \text{ wk. } 3 \text{ da. } 17 \text{ hr.} \\ - (2 \text{ wk. } 6 \text{ da. } 20 \text{ hr.}) \\ \hline \end{array}$
6.	$\begin{array}{r} 17 \text{ yd. } 2 \text{ ft. } 11 \text{ in.} \\ + (26 \text{ yd. } 2 \text{ ft. } 9 \text{ in.}) \\ \hline \end{array}$
7.	$\begin{array}{r} 9 \text{ hr. } 17 \text{ min. } 42 \text{ sec.} \\ - (6 \text{ hr. } 32 \text{ min. } 57 \text{ sec.}) \\ \hline \end{array}$
8.	$\begin{array}{r} 16 \text{ m. } 8 \text{ dm. } 3 \text{ cm. } 5 \text{ mm.} \\ - (12 \text{ m. } 8 \text{ dm. } 7 \text{ cm. } 6 \text{ mm.}) \\ \hline \end{array}$

Perimeter, Area



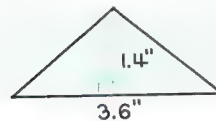
$$\begin{aligned} p &= 2(l + w) \\ &= 2(3.8 + 1.7) \\ &= 2(5.5) \\ &= 11 \end{aligned}$$

Perimeter is 11 inches.



$$\begin{aligned} A &= lw \\ &= 8 \times 7 \\ &= 56 \end{aligned}$$

Area is 56 square inches.

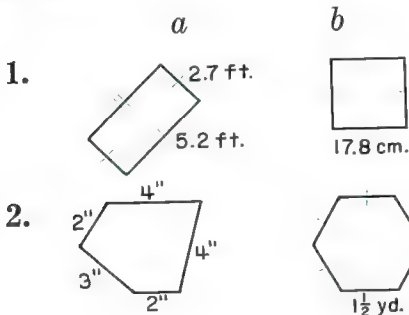


$$\begin{aligned} A &= \frac{1}{2}ab \\ &= \frac{1}{2}(1.4 \times 3.6) \\ &= 2.52 \end{aligned}$$

Area is 2.52 square inches.

Use after page 271.

Find the perimeter of each polygon below.



Find the perimeter of each rectangle described below.

<i>a</i>		<i>b</i>	
<i>Length, Width</i>		<i>Length, Width</i>	
3. 9 yd., 4 yd.		12.2 m., 5.8 m.	
4. $3\frac{1}{8}$ in., $2\frac{5}{8}$ in.		103 ft., 67 ft.	
5. .9 ft., .7 ft.		1.45 in., .98 in.	
6. $7\frac{1}{4}$ in., $6\frac{3}{4}$ in.		1.17 cm., 1.17 cm.	

Use after page 279.

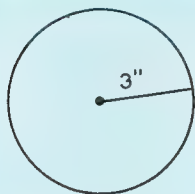
Find the area of each rectangle described below.

- Length 17 cm., width 13 cm.
- Length $7\frac{1}{2}$ ft., width $4\frac{1}{2}$ ft.
- Length 4.6 in., width 2.8 in.
- Length $62\frac{1}{6}$ yd., width $21\frac{5}{6}$ yd.

Find the area of each triangle described below.

- Base 17 yd., altitude 8 yd.
- Base 3.5 ft., altitude 7.1 ft.
- Base $5\frac{3}{8}$ in., altitude $4\frac{5}{8}$ in.
- Base 24 cm., altitude 24 cm.
- Base 12.4 ft., altitude 9.6 ft.
- Base $11\frac{1}{2}$ in., altitude $8\frac{1}{2}$ in.

Circumference, Area, Volume



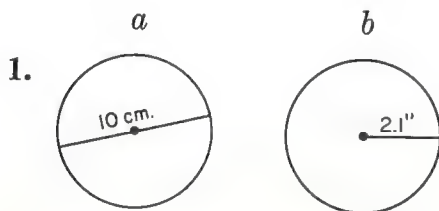
$$\begin{aligned} C &= 2\pi r \\ &= 2 \times 3.14 \times 3 \\ &= 18.84 \\ A &= \pi r^2 \\ &= 3.14 \times 3^2 \\ &= 28.26 \end{aligned}$$

Circumference is 18.84 inches.

Area is 28.26 square inches.

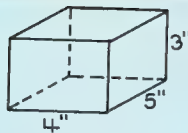
Use after page 287.

Find the circumference and the area of each circle below. Use 3.14 for π .



Find the circumference and the area of each circle described below. Use 3.14 for π .

- | a | b |
|-------------------|------------------|
| 2. Radius 7 yd. | Diameter 20 mi. |
| 3. Radius 1 ft. | Diameter 16 in. |
| 4. Radius 6 in. | Diameter 18 ft. |
| 5. Radius 1.5 in. | Diameter 4.8 cm. |
| 6. Radius 12 cm. | Diameter 100 in. |
| 7. Radius 40 mi. | Diameter 21 ft. |



$$\begin{aligned} V &= lwh \\ &= 4 \times 5 \times 3 \\ &= 60 \end{aligned}$$

Volume is 60 cubic inches.



$$\begin{aligned} V &= \pi r^2 h \\ &= 3.14 \times 2^2 \times 7 \\ &= 3.14 \times 4 \times 7 \\ &= 87.92 \end{aligned}$$

Volume is 87.92 cubic inches.

Use after page 290.

Find the volume of each rectangular solid described below.

	<i>Length</i>	<i>Width</i>	<i>Height</i>
1.	12 in.	8 in.	5 in.
2.	$3\frac{1}{8}$ in.	$6\frac{5}{8}$ in.	$4\frac{3}{8}$ in.
3.	4.5 ft.	2.6 ft.	1.7 ft.
4.	12 cm.	8 cm.	6 cm.
5.	20 yd.	17 yd.	3 yd.

Find the volume of each circular cylinder described below. Use 3.14 for π .

- Radius 3 in., height 8 in.
- Radius 1 ft., height 3.2 ft.
- Radius 4 m., height 10 m.
- Radius 7 in., height 3 in.

Organizing Data

Set of data:

{5, 8, 8, 9, 11, 13}

Mode: 8

Median: $\frac{8+9}{2}$ or 8.5

Mean: $\frac{5+8+8+9+11+13}{6}$ or 9

Use after page 297.

Find the mode, the median, and the mean for each set of data given below.

1. {5, 5, 4, 8, 6, 11, 17, 3, 4}

2. $\{2\frac{1}{2}, 4\frac{3}{8}, 1\frac{1}{4}, 3\frac{5}{8}, 2\frac{1}{2}\}$

3. {2.6, 3.8, .7, 5.4, 1.5}

4. {17, 13, 17, 10, 17, 10}

5. {52, 26, 18, 63, 48, 63, 17}

6. $\{1\frac{1}{2}, 6\frac{1}{4}, 3\frac{3}{8}, 1\frac{1}{2}, 5\frac{1}{16}, 1\frac{1}{2}\}$

A histogram shows the distribution of frequencies within a set of data.

Use after page 301.

Make a histogram for each of the following sets of data.

1. Number of desks in different classrooms

Desks	22	23	24	25	26	27
Frequency	2	3	4	4	2	1

2. Number of words per line

14, 16, 16, 12, 15, 14, 15, 13, 15, 14, 15, 13, 15, 16, 16, 17, 12, 15, 14, 16, 15, 13, 14, 16, 17, 15, 16, 15, 14, 17

3. Number of miles per gallon of gasoline for automobiles

17, 19, 15, 21, 19, 20, 18, 18, 21, 18, 21, 18, 19, 16, 17, 20, 22, 19, 18, 15, 17, 19, 18, 17, 20



The probability of drawing a blue marble on the first draw is 4 out of 7 or $\frac{4}{7}$.

Use after page 309.

Use the set of marbles shown above to find the probability of drawing marbles as described below.

1. black marble on first draw

2. first a black; replace it; then a blue

3. first a blue; do not replace it; then another blue

4. first a black; do not replace it; then a blue

5. first a blue; replace it; then another blue

Diagnostic Self-Tests

The tests on pages 335–338 will help you and your teacher to see how much you have learned in mathematics. Do each test and check your work before you give your paper to your teacher.

Self-Test 1—Numeration

Write a Roman numeral for each base-ten numeral below.

<i>a</i>	<i>b</i>	<i>c</i>
1. 34	169	1420
2. 49	328	2644
3. 99	532	3428

Find the simplest base-ten numeral for each number below.

<i>a</i>	<i>b</i>	<i>c</i>
4. 32 _{five}	41 _{seven}	101 _{two}
5. 44 _{five}	65 _{seven}	110 _{two}
6. 120 _{five}	236 _{seven}	1111 _{two}

Change each base-ten numeral below to a numeral in the base that is indicated.

<i>a</i>	<i>b</i>
7. $32 = \underline{\hspace{1cm}}$ _{five}	$140 = \underline{\hspace{1cm}}$ _{five}
8. $64 = \underline{\hspace{1cm}}$ _{seven}	$352 = \underline{\hspace{1cm}}$ _{seven}
9. $10 = \underline{\hspace{1cm}}$ _{two}	$75 = \underline{\hspace{1cm}}$ _{two}

Self-Test 2—Mathematical Sentences

Insert () and [] in each equation so that it becomes true.

- $48 - 36 \div 6 \times 2 = 45$
- $48 - 36 \div 6 \times 2 = 36$
- $48 - 36 \div 6 \times 2 = 4$
- $48 - 36 \div 6 \times 2 = 84$

Replace each \odot with =, <, or > so that the resulting sentence is true.

- | <i>a</i> | <i>b</i> |
|--|-----------------------------|
| 5. $5 \times 8 \odot 38$ | $(12 - 4) \div 2 \odot 7$ |
| 6. $\frac{1}{3} \times \frac{2}{5} \odot \frac{2}{15}$ | $96 \div 12 \odot 41 - 33$ |
| 7. $17 \odot 32 - 8$ | $3 \odot 19 - (4 \times 3)$ |

Use {1, 2, 3, 4, 5, 6, 7, 8, 9} as the replacement set to solve each open sentence below.

- | <i>a</i> | <i>b</i> |
|------------------|-----------------|
| 8. $34 + a = 41$ | $3b + 2 < 12$ |
| 9. $5n < 21$ | $28 \div c = 4$ |
| 10. $x - 3 > 0$ | $7n > 82 - 7$ |

Self-Test 3—Addition

Solve each equation below.

- | a | b |
|--------------------------------------|--|
| 1. $385 + 226 = n$ | $5281 + 397 = n$ |
| 2. $-7 + +2 = x$ | $a = (-3 + +3) + -9$ |
| 3. $-5 + -8 = y$ | $b = +17 + (-8 + +5)$ |
| 4. $\frac{3}{7} + \frac{2}{7} = c$ | $r = \frac{2}{3} + (\frac{1}{4} + \frac{5}{6})$ |
| 5. $\frac{3}{8} + \frac{1}{4} = f$ | $s = 3\frac{1}{2} + (\frac{1}{3} + \frac{1}{6})$ |
| 6. $7\frac{3}{4} + 2\frac{5}{6} = g$ | $t = 2\frac{3}{8} + 1\frac{3}{5} + 8\frac{1}{2}$ |
| 7. $.7 + .5 = h$ | $a = 1.32 + 3.6$ |
| 8. $.73 + .8 = i$ | $n = 5 + 3.26 + .09$ |

Copy. Find the simplest numeral for each sum.

- | a | b |
|--|---|
| 9. $\begin{array}{r} 5296 \\ +1739 \\ \hline \end{array}$ | $\begin{array}{r} 28.346 \\ +9.27 \\ \hline \end{array}$ |
| 10. $\begin{array}{r} 236\frac{5}{8} \\ +129\frac{1}{6} \\ \hline \end{array}$ | $\begin{array}{r} 321.52 \\ +548.658 \\ \hline \end{array}$ |

Express each sum with the least number of each kind of unit.

11.
$$\begin{array}{r} 38 \text{ hr. } 26 \text{ min. } 54 \text{ sec.} \\ + (12 \text{ hr. } 38 \text{ min. } 17 \text{ sec.}) \\ \hline \end{array}$$
12.
$$\begin{array}{r} 17 \text{ yd. } 1 \text{ ft. } 10 \text{ in.} \\ 8 \text{ yd. } 2 \text{ ft. } 7 \text{ in.} \\ + (10 \text{ yd. } 2 \text{ ft. } 8 \text{ in.}) \\ \hline \end{array}$$

Self-Test 4—Subtraction

Solve each equation.

- | a | b |
|---------------------------------------|-------------------------------------|
| 1. $-8 - +3 = a$ | $+9 - -6 = n$ |
| 2. $+3 - +11 = b$ | $m = -12 - -8$ |
| 3. $c = \frac{5}{8} - \frac{3}{8}$ | $r = \frac{3}{4} - \frac{3}{8}$ |
| 4. $d = 7 - \frac{5}{6}$ | $s = 8\frac{3}{4} - 5\frac{1}{4}$ |
| 5. $5\frac{7}{12} - 2\frac{5}{6} = e$ | $19\frac{1}{8} - 12\frac{3}{5} = t$ |
| 6. $3.8 - .4 = f$ | $38.27 - 5.9 = n$ |

Copy. Find the simplest numeral for each difference. Express all fractions and mixed numerals in simplest form.

- | a | b | c |
|--|--|---|
| 7. $\begin{array}{r} 738 \\ -526 \\ \hline \end{array}$ | $\begin{array}{r} 4705 \\ -677 \\ \hline \end{array}$ | $\begin{array}{r} 6426 \\ -3789 \\ \hline \end{array}$ |
| 8. $\begin{array}{r} 54.1 \\ -26.7 \\ \hline \end{array}$ | $\begin{array}{r} 4.73 \\ -2.586 \\ \hline \end{array}$ | $\begin{array}{r} 4 \\ -1.3081 \\ \hline \end{array}$ |
| 9. $\begin{array}{r} 18\frac{7}{8} \\ -8\frac{3}{8} \\ \hline \end{array}$ | $\begin{array}{r} 26\frac{3}{7} \\ -19\frac{2}{5} \\ \hline \end{array}$ | $\begin{array}{r} 148\frac{1}{4} \\ -89\frac{4}{5} \\ \hline \end{array}$ |

Express each difference with the least number of each kind of unit.

10.
$$\begin{array}{r} 5 \text{ wk. } 3 \text{ da. } 17 \text{ hr. } 21 \text{ min.} \\ - (2 \text{ wk. } 5 \text{ da. } 19 \text{ hr. } 46 \text{ min.}) \\ \hline \end{array}$$
11.
$$\begin{array}{r} 4 \text{ m. } 3 \text{ dm. } 4 \text{ cm. } 7 \text{ mm.} \\ - (3 \text{ m. } 7 \text{ dm. } 6 \text{ cm. } 9 \text{ mm.}) \\ \hline \end{array}$$

Self-Test 5—Multiplication

Copy. Find the simplest numeral for each product.

	<i>a</i>	<i>b</i>	<i>c</i>
1.	$\begin{array}{r} 38 \\ \times 7 \\ \hline \end{array}$	$\begin{array}{r} 532 \\ \times 24 \\ \hline \end{array}$	$\begin{array}{r} 4302 \\ \times 302 \\ \hline \end{array}$
2.	$\begin{array}{r} 59 \\ \times 86 \\ \hline \end{array}$	$\begin{array}{r} 485 \\ \times 356 \\ \hline \end{array}$	$\begin{array}{r} 7038 \\ \times 6464 \\ \hline \end{array}$
3.	$\begin{array}{r} 3.6 \\ \times .8 \\ \hline \end{array}$	$\begin{array}{r} 1.26 \\ \times 4.3 \\ \hline \end{array}$	$\begin{array}{r} 3.28 \\ \times 26.4 \\ \hline \end{array}$
4.	$\begin{array}{r} .89 \\ \times .7 \\ \hline \end{array}$	$\begin{array}{r} 3.47 \\ \times 2.06 \\ \hline \end{array}$	$\begin{array}{r} .542 \\ \times 7.35 \\ \hline \end{array}$

Copy. Solve each equation. Express all answers in simplest form.

	<i>a</i>	<i>b</i>
5.	$47 \times 8 = a$	$f = 17 \times 256$
6.	$5 \times \frac{2}{3} = n$	$r = (\frac{3}{4} \times \frac{4}{7}) \times \frac{2}{3}$
7.	$\frac{2}{5} \times \frac{1}{3} = m$	$t = \frac{4}{15} \times (\frac{10}{12} \times \frac{7}{8})$
8.	$3\frac{1}{2} \times \frac{3}{4} = a$	$s = 7\frac{3}{5} \times 5\frac{1}{4} \times 3\frac{1}{2}$
9.	$.7 \times 10 = b$	$1.7 \times 100 = q$
10.	$39 \times 7 = c$	$3.9 \times .7 = k$
11.	$352 \times 100 = a$	$.352 \times 100 = n$
12.	$5.2 \times .3 = r$	$t = .92 \times .01$
13.	$-5 \times -7 = q$	$x = -17 \times +6$
14.	$+8 \times -4 = f$	$y = +21 \times +13$

Self-Test 6—Division

Copy. Find the simplest numeral for each quotient and remainder.

	<i>a</i>	<i>b</i>
1.	$4 \overline{)616}$	$7 \overline{)1582}$
2.	$34 \overline{)1768}$	$47 \overline{)6298}$
3.	$58 \overline{)2053}$	$408 \overline{)53856}$
4.	$226 \overline{)3842}$	$65 \overline{)8507}$
5.	$314 \overline{)5672}$	$524 \overline{)84027}$

Copy. Solve each equation.

	<i>a</i>	<i>b</i>
6.	$\frac{7}{8} \div \frac{3}{8} = n$	$m = \frac{3}{5} \div \frac{1}{4}$
7.	$\frac{14}{25} \div \frac{7}{15} = x$	$y = \frac{6}{7} \div \frac{132}{147}$
8.	$3\frac{2}{3} \div \frac{5}{6} = a$	$b = 7\frac{1}{2} \div 2\frac{2}{3}$
9.	$4\frac{1}{5} \div 7 = c$	$d = \frac{6}{7} \div 9\frac{3}{5}$
10.	$15\frac{1}{3} \div 4\frac{3}{5} = k$	$r = 7\frac{3}{8} \div 7\frac{3}{8}$

Copy. Find each quotient.

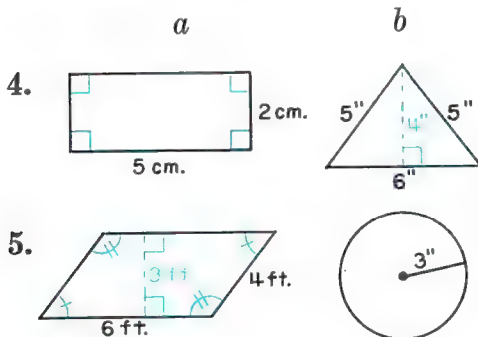
	<i>a</i>	<i>b</i>
11.	$8 \overline{)11.2}$	$24 \overline{)77.04}$
12.	$.7 \overline{)3.15}$	$4.3 \overline{)5.461}$
13.	$.26 \overline{)6.474}$	$3.07 \overline{)1.26484}$
14.	$.86 \overline{)20.124}$	$1.58 \overline{)2.4964}$

Self-Test 7—Geometry

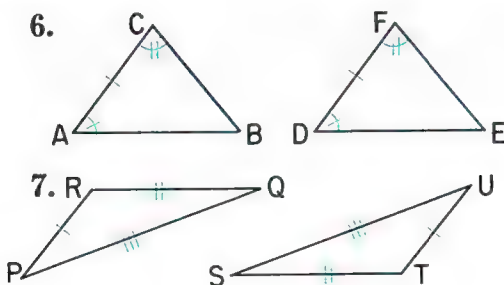
Draw and label each of the following figures.

- | <i>a</i> | <i>b</i> |
|--------------|-----------------|
| 1. line AB | line segment RS |
| 2. ray DE | triangle PQR |
| 3. angle ABC | rectangle EFGH |

Find the perimeter and the area of each figure below.



Write 3 statements about congruent angles or congruent sides for each pair of triangles below. Then state a reason why the two triangles are congruent.



Self-Test 8—Ratio, Per Cent

Solve each of the following proportions.

- | <i>a</i> | <i>b</i> | <i>c</i> |
|----------------------------------|--------------------------------|--------------------------------|
| 1. $\frac{5}{8} = \frac{n}{64}$ | $\frac{n}{15} = \frac{14}{35}$ | $\frac{15}{n} = \frac{35}{63}$ |
| 2. $\frac{6}{14} = \frac{24}{n}$ | $\frac{n}{48} = \frac{75}{90}$ | $\frac{21}{n} = \frac{42}{48}$ |

Copy. Complete the table so that all the numerals in each row name the same number.

	Fraction	Decimal	Per cent
3.	$\frac{4}{5}$	—	—
4.	—	.75	—
5.	—	—	95%
6.	—	1.36	—
7.	—	—	225%
8.	$3\frac{7}{20}$	—	—

Answer each of the following questions.

9. What number is 25% of 96?

10. 21 is what per cent of 35?

11. 12 is 15% of what number?

12. A baseball team lost 12 of the 60 games they played. What is the ratio of the number of wins to the number of losses? What per cent of the games did they win?

HANDBOOK

Many of your questions about mathematics can be answered by using this handbook. An example is given for each idea or computation.

The handbook is easy to use. Its twelve sections are arranged in alphabetical order. The parts of each section are listed in the order they appear throughout the book. The numerals in () tell the pages in the book that give more information on that topic.

Addition

Terms and Concepts

1. Identity number Zero is the identity number of addition. For every number n , $n+0=n=0+n$. (52, 95, 144)

$$\begin{array}{lll} 0+17=17 & -5+0=-5 & 0+\frac{5}{7}=\frac{5}{7} \\ 21+0=21 & 0+6=6 & \frac{2}{3}+0=\frac{2}{3} \end{array}$$

2. Commutative property For any two numbers a and b , $a+b=b+a$. (52, 96, 144)

Whole numbers: $17+32=32+17$

Integers: $-3+5=5-3$

Rational numbers: $\frac{2}{3}+\frac{4}{11}=\frac{4}{11}+\frac{2}{3}$

3. Associative property For any numbers a , b , and c , $(a+b)+c=a+(b+c)$. (53, 96, 144)

Whole numbers: $(5+18)+72=5+(18+72)$

Integers: $(-1+5)+-8=-1+(5-8)$

Rational numbers: $(\frac{2}{3}+\frac{3}{4})+\frac{5}{7}=\frac{2}{3}+(\frac{3}{4}+\frac{5}{7})$

4. Inverse operation If $a+b=c$, then $c-b=a$. (54, 98, 146)

$$7+8=15 \text{ so } 15-8=7$$

How to Add Integers

1. Two positive integers The sum of two positive integers is a positive integer. (94-97)

$$+6+7=+13$$

2. Two negative integers The sum of two negative integers is a negative integer. (94-97)

$$-8+-4=-12$$

3. A positive and a negative integer The coordinate farther from the 0 mark on a number line determines whether the sum is a positive or a negative integer. (94)

$$-9+6=-3$$

$$-5+7=+2$$

How to Add Rational Numbers

1. With the same denominator If $\frac{a}{n}$ and $\frac{b}{n}$ represent rational numbers, then $\frac{a}{n} + \frac{b}{n} = \frac{a+b}{n}$. (122)

$$\frac{6}{13} + \frac{3}{13} = \frac{6+3}{13} = \frac{9}{13}$$

2. With different denominators If the denominators are not the same, rename the numbers so that the denominators are the same. Then add as follows. (140–143)

$$\begin{array}{r} \frac{2}{5} + \frac{1}{4} = \frac{8}{20} + \frac{5}{20} \\ = \frac{8+5}{20} \\ = \frac{13}{20} \end{array} \qquad \begin{array}{r} \frac{2}{5} \\ + \frac{1}{4} \\ \hline \frac{13}{20} \end{array} \qquad \begin{array}{r} \frac{8}{20} \\ \frac{5}{20} \\ \hline \frac{13}{20} \end{array}$$

3. Least common denominator The least common denominator of two or more rational numbers is the least common multiple of their denominators. (138, 140)

$$\begin{aligned} \frac{1}{6} + \frac{3}{4} &= \frac{2}{12} + \frac{9}{12} \\ &= \frac{2+9}{12} \\ &= \frac{11}{12} \end{aligned}$$

4. Simplest form A fraction is in simplest form if the numerator and the denominator are relatively prime. (118)

$$\frac{8}{10} = \frac{8 \div 2}{10 \div 2} = \frac{4}{5}$$

A mixed numeral is in simplest form if the fraction is in simplest form and names a number between 0 and 1. (123)

$$7\frac{8}{6} = 7 + \frac{6}{6} + \frac{2}{6} = 8\frac{2}{6} = 8\frac{1}{3}$$

5. With mixed numerals When adding with mixed numerals, add the fractional numbers and add the whole numbers. (123, 140)

$$\begin{array}{r} n = 1\frac{1}{3} + 2\frac{1}{2} \\ = (1 + \frac{2}{6}) + (2 + \frac{3}{6}) \\ = (1 + 2) + (\frac{2}{6} + \frac{3}{6}) \\ = 3 + \frac{5}{6} \\ = 3\frac{5}{6} \end{array} \qquad \begin{array}{r} 1\frac{1}{3} \\ + 2\frac{1}{2} \\ \hline 3\frac{5}{6} \end{array} \qquad \begin{array}{r} 1\frac{2}{6} \\ 2\frac{3}{6} \\ \hline 3\frac{5}{6} \end{array}$$

6. With decimals Add with decimal numerals just as you add with fractions. (164)

$$\begin{aligned} \frac{3}{4} + \frac{2}{5} &= \frac{15}{20} + \frac{8}{20} = \frac{23}{20} = 1\frac{3}{20} \\ &= 1.15 \end{aligned}$$

7. In decimal form Write the decimals so that the decimal points align vertically. Then add as you do when the addends are whole numbers. (164)

$$\begin{array}{r} 183.015 \\ 76.964 \\ + 217.218 \\ \hline 477.197 \end{array}$$

8. Decimals that denote different place values The decimals 3.5, 3.50, and 3.500 all name the same number. Hence, any of them can be used to name the same addend. (164)

$$\begin{array}{r} 7.283 \\ 3.5 \\ + 2.48 \\ \hline 13.263 \end{array} \qquad \begin{array}{r} 7.283 \\ 3.50 \\ + 2.48 \\ \hline 13.263 \end{array} \qquad \begin{array}{r} 7.283 \\ 3.500 \\ + 2.48 \\ \hline 13.263 \end{array}$$

Decimals

Terms and Concepts

1. Decimal numeral A decimal numeral is a base-ten numeral that names either a whole number or a rational number. (155)

237 5.42 .016

2. Terminating decimal If the denominator of a fraction in simplest form has prime factors of only 2, only 5, or only 2 and 5, it can be expressed as a terminating decimal. (160-163)

$$\frac{9}{20} = \frac{9}{20} \times \frac{5}{5} = \frac{45}{100} \text{ or } .45$$

3. Repeating decimal If the denominator of a fraction in simplest form has a prime factor other than 2 or 5, the fraction can be expressed as a repeating decimal. (180)

$$\frac{2}{3} = .666 \dots \text{ or } \overline{.6}$$

How to Change a Decimal to a Fraction

Reading a decimal Reading a decimal enables you to change it to a fraction or to a mixed numeral. Tenths, hundredths, and so on refer to fractions or to decimals. (158)

15.08

fifteen and eight hundredths

$$15\frac{8}{100} \text{ or } 15\frac{2}{25}$$

How to Change a Fraction to a Decimal

1. To a terminating decimal

- a. Determine the least power of ten that has the denominator as a factor. Then use the relationship between equivalent fractions and solve the proportion. (160)

$$\begin{aligned} \frac{3}{4} &= \frac{x}{100} \text{ so } 3 \times 100 = 4x \\ \frac{3 \times 100}{4} &= x \\ 3 \times 25 &= x \\ 75 &= x \end{aligned}$$

$$\text{Therefore, } \frac{3}{4} = \frac{75}{100} \text{ or } .75.$$

- b. Divide the numerator by the denominator. (178)

$$\begin{array}{r} \frac{3}{4} \qquad .75 \\ 4 \overline{) 3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

- 2. To a repeating decimal** Divide the numerator by the denominator until you observe a repeating difference. (180-181)

$$\begin{array}{r} \frac{7}{11} \qquad .63 \\ 11 \overline{) 7.00} \\ \underline{66} \\ 40 \\ \underline{33} \\ 7 \end{array}$$

$$\text{Therefore, } \frac{7}{11} = \overline{.63}.$$

Distributive Properties

Terms and Concepts

1. Multiplication distributes over addition For all numbers a , b , and c , $a(b+c) = (a \times b) + (a \times c)$ and also $(b+c)a = (b \times a) + (c \times a)$. (62, 102, 124)

Whole numbers: $5(6+7) = (5 \times 6) + (5 \times 7)$

Integers: $-3(+4 + -1) = (-3 \times +4) + (-3 \times -1)$

Rational numbers: $\frac{2}{3} \times 6\frac{1}{2} = (\frac{2}{3} \times 6) + (\frac{2}{3} \times \frac{1}{2})$

2. Division distributes over addition Division can be distributed over addition if the dividend is named as a sum. Use addends that are multiples of the divisor. (71)

$$\begin{aligned} 72 \div 6 &= (60 + 12) \div 6 \\ &= (60 \div 6) + (12 \div 6) \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

Division

Terms and Concepts

1. Inverse operation For all numbers a , b , and c (except $b=0$), if $a \times b = c$, then $c \div b = a$. (66, 104)

$$3 \times 6 = 18 \text{ so } 18 \div 6 = 3$$

2. Zero in division If zero is divided by any natural number, the quotient is zero. (68)

$$0 \div 3 = 0$$

$$0 \div 17 = 0$$

Division by zero is meaningless. Zero is never used as a divisor. (68)

$7 \div 0$ is meaningless.

3. Divisible by Let a and b represent whole numbers and $a \neq 0$. If a is a factor of b , then b is divisible by a . (69, 78–80)

$$15 \div 3 = 5$$

15 is divisible by 3.

How to Divide Whole Numbers

1. Use the distributive property Rename the dividend as a sum and distribute division over addition. (71–73)

$$\begin{aligned} 536 \div 4 &= (400 + 120 + 16) \div 4 \\ &= (400 \div 4) + (120 \div 4) + (16 \div 4) \\ &= 100 + 30 + 4 \\ &= 134 \end{aligned}$$

2. Use the division algorithm Estimate each digit of the quotient by using inequalities. Revise the estimate if necessary. (74–77)

$$\begin{array}{r} 124 \\ 54 \overline{) 6696} \\ \underline{54} \\ 129 \\ \underline{108} \\ 216 \\ \underline{216} \\ 0 \end{array}$$

$$7\text{Th} \div 5\text{T} \geq 1\text{H}$$

$$13\text{H} \div 5\text{T} \geq 2\text{T}$$

$$22\text{T} \div 5\text{T} \geq 4$$

How to Divide Integers

1. Two positive integers The quotient of two positive integers is a positive number. (104)

$$+12 \div +4 = +3 \qquad +5 \div +2 = +\frac{5}{2}$$

2. Two negative integers The quotient of two negative integers is a positive number. (104)

$$-8 \div -4 = +2 \qquad -7 \div -3 = +\frac{7}{3}$$

3. A positive integer and a negative integer The quotient of two integers, one positive and the other negative, is a negative number. (104)

$$-15 \div +5 = -3 \qquad +9 \div -4 = -\frac{9}{4}$$

How to Divide Rational Numbers

1. Reciprocals If the product of two numbers is 1, the numbers are called reciprocals of each other. (126)

$$\frac{7}{9} \times \frac{9}{7} = 1$$

$\frac{7}{9}$ is the reciprocal of $\frac{9}{7}$.

$\frac{9}{7}$ is the reciprocal of $\frac{7}{9}$.

2. Using reciprocals Dividing by a rational number is equivalent to multiplying by its reciprocal. For rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. (126)

$$\frac{7}{8} \div \frac{3}{5} = \frac{7}{8} \times \frac{5}{3} = \frac{35}{24} = 1\frac{11}{24}$$

$$\frac{5}{6} \div 10 = \frac{5}{6} \times \frac{1}{10} = \frac{1}{12}$$

3. With decimals The procedures for dividing whole numbers can be extended to division with decimals.

a. Whole number divisor Estimate each quotient digit as you do for whole numbers. (172)

$$\begin{array}{r} .27 \\ 4 \overline{) 1.08} \\ \underline{8} \\ 28 \\ \underline{28} \\ 0 \end{array}$$

$$11\text{Ts} \div 4 \geq 2\text{Ts}$$

$$28\text{Hs} \div 4 \geq 7\text{Hs}$$

b. Changing to a whole number divisor If the divisor is named by a decimal, the division expression can be changed to obtain a whole number divisor. (176)

$$\begin{aligned} .28 \div .7 &= \frac{.28}{.7} \\ &= \frac{.28}{.7} \times 1 \\ &= \frac{.28}{.7} \times \frac{10}{10} \\ &= \frac{2.8}{7} \end{aligned}$$


c. Using the division algorism Change the division expression so that the divisor is a whole number. Then divide as you divide whole numbers. (176)

$$\begin{array}{r} .235 \\ .47 \overline{) 11.045} \\ \underline{94} \\ 164 \\ \underline{141} \\ 235 \\ \underline{235} \\ 0 \end{array}$$


Geometry

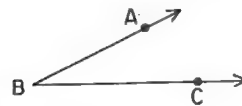
Terms and Concepts

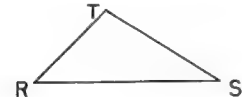
Use symbols to represent geometric ideas The drawings and symbols below are representations of geometric ideas.


a. 
 \overleftrightarrow{CD} means *line* CD. (207)

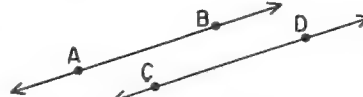
b. 
 \overline{CD} means *line segment* CD. (208)

c. 
 \overrightarrow{CD} means *ray* CD. (208)

d. 
 $\angle ABC$ means *angle* ABC. (214)

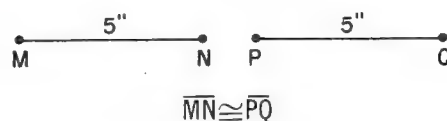
e. 
 $\triangle RST$ means *triangle* RST. (232)

f. 
 $\overline{AB} \perp \overline{CD}$ means \overline{AB} is *perpendicular* to \overline{CD} . (227)

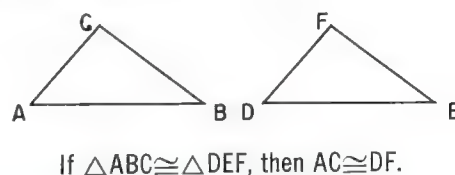
g. 
 $\overline{AB} \parallel \overline{CD}$ means \overline{AB} is *parallel* to \overline{CD} . (244)

How to Determine Congruent Line Segments

1. Find their measures Two line segments are congruent if they have the same measure. (212)

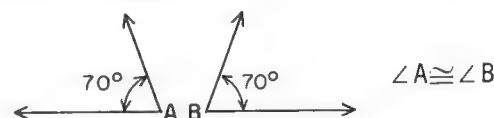


2. Corresponding parts of congruent figures If two figures are congruent, their corresponding parts are congruent. (234–236)

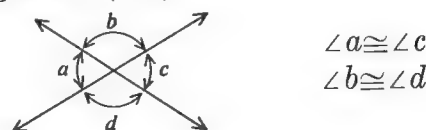


How to Determine Congruent Angles

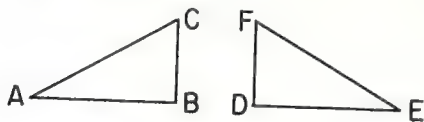
1. Find their measures Two angles are congruent if they have the same measure. (218)



2. Vertical angles If two lines intersect, the vertical angles are congruent. (222)

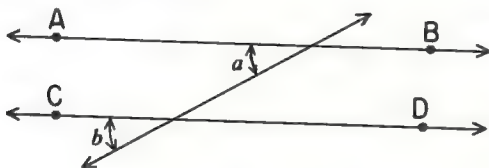


3. Corresponding parts of congruent figures If two polygons are congruent, their corresponding angles are congruent. (234-236)



If $\triangle ABC \cong \triangle EDF$, then $\angle A \cong \angle E$.

4. With parallel lines Corresponding angles on parallel lines are congruent. (244)



If $\overline{AB} \parallel \overline{CD}$, then $\angle a \cong \angle b$.

How to Classify Triangles

1. According to sides Triangles are classified according to the number of congruent sides. (232)



scalene



isosceles



equilateral

2. According to angles Triangles are classified according to the kinds of angles they contain. (232)



acute



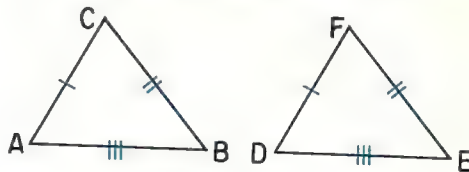
obtuse



right

How to Determine Congruent Triangles

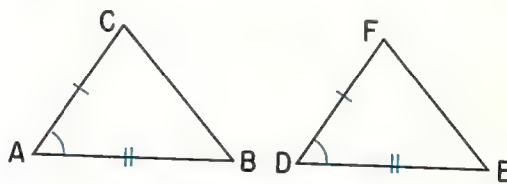
1. Three sides of one congruent to three sides of the other (234)



$\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$

$\triangle ABC \cong \triangle DEF$

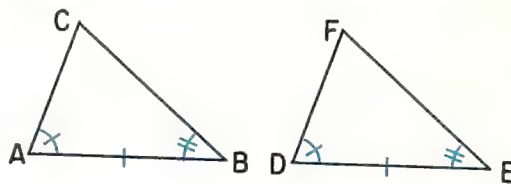
2. Two sides and the included angle of one congruent to two sides and the included angle of the other (236)



$\overline{AC} \cong \overline{DF}$, $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$

$\triangle ABC \cong \triangle DEF$

3. Two angles and the included side of one congruent to two angles and the included side of the other (237)



$\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$

$\triangle ABC \cong \triangle DEF$

Mathematical Sentences

Terms and Concepts

1. Grouping symbols The () and [] are common grouping symbols. They clarify the meaning of mathematical expressions and indicate the order of operations. (30)

$$\begin{aligned} &[(36 \div 6) \times 2] + 7 \\ &[6 \times 2] + 7 \\ &12 + 7 \\ &19 \end{aligned}$$

2. Variable A variable or a placeholder is a letter or some other symbol that holds the place for a numeral. (32)

x is a variable in $3x + 2$.

3. Equality sentence An equality sentence or an equation contains the symbol =. (34)

$$5x + 2 = 8 \qquad 17 - 8 = 9$$

4. Inequality sentence An inequality sentence or an inequality contains one of the symbols \neq , $<$, or $>$. (36)

$$36 \neq 9 + 8 \qquad 7a < 32$$

5. Properties of equality Equality has the following properties for all numbers a , b , and c . (48)

$$a = a$$

$$\text{If } a = b, \text{ then } b = a.$$

$$\text{If } a = b \text{ and } b = c, \text{ then } a = c.$$

How to Solve Open Sentences

1. Equality sentences Find all the members of the replacement set that make the open sentence become a true closed sentence. (38)

$$\text{Replacement set: } \{3, 4, 5, 6, 7, 8, 9\}$$

$$\text{Open sentence: } 5x = 30$$

$$\text{Solution set: } \{6\}$$

2. Inequality sentences Find all the members of the replacement set that make the open sentence become a true closed sentence. (40)

$$\text{Replacement set: } \{0, 1, 2, 3, 4, 5, 6\}$$

$$\text{Open sentence: } 7 + n < 12$$

$$\text{Solution set: } \{0, 1, 2, 3, 4\}$$

How to Evaluate Expressions

1. With one variable Replace the variable by the given value. (44)

If $n = 7$, then

$$\begin{aligned} 5n + 8 &= (5 \times 7) + 8 \\ &= 35 + 8 \\ &= 43 \end{aligned}$$

2. Same variable used more than once Replace each occurrence of the variable with a name for the same number. (44)

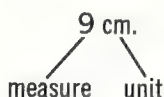
If $x = 12$, then

$$\begin{aligned} 6x + (48 \div x) &= (6 \times 12) + (48 \div 12) \\ &= 72 + 4 \\ &= 76 \end{aligned}$$

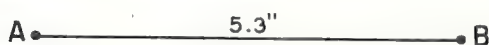
Measurement

Terms and Concepts

1. A measurement A measurement consists of two symbols—a numeral for the measure and a word, abbreviation, or symbol for the unit. (255)



2. Measurement is approximate We can measure only to the nearest unit of measurement. (255)



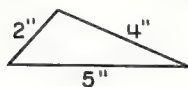
\overline{AB} is 5.3" long, to the nearest .1 inch.

3. Range of measurement The range of a measurement is the set of all measurements from $\frac{1}{2}$ unit less to $\frac{1}{2}$ unit more than the given measurement. (256)

Measurement: $17\frac{1}{2}$ ft.

Range of measurement: $17\frac{1}{4}$ ft. to $17\frac{3}{4}$ ft.

4. Perimeter The perimeter of a simple closed figure is the distance around the figure. (269-271)



$$p = 2 + 4 + 5 = 11$$

Perimeter is 11 inches.

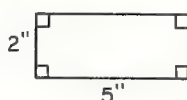


$$p = 2(l + w)$$

$$= 2(3 + 5) \text{ or } 16$$

Perimeter is 16 inches.

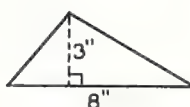
5. Area Area is the measurement of the interior of a simple closed figure. (272-281, 286)



$$A = lw$$

$$= 2 \times 5 = 10$$

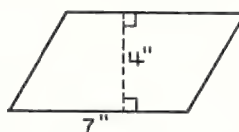
Area is 10 sq. in.



$$A = \frac{1}{2}ab$$

$$= \frac{1}{2}(3 \times 8) = 12$$

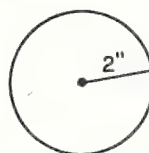
Area is 12 sq. in.



$$A = ab$$

$$= 4 \times 7 = 28$$

Area is 28 sq. in.



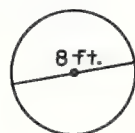
$$A = \pi r^2$$

$$= 3.14 \times 2^2$$

$$= 3.14 \times 4 = 12.56$$

Area is 12.56 sq. in.

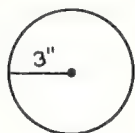
6. Circumference The perimeter of a circle is called its circumference. (282-285)



$$C = \pi d$$

$$= 3.14 \times 8 = 25.12$$

Circumference is 25.12 ft.

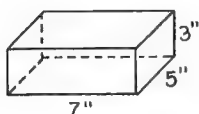


$$C = 2\pi r$$

$$= 2 \times 3.14 \times 3 = 18.84$$

Circumference is 18.84 in.

7. Volume Volume is the measurement of the interior of a solid or 3-dimensional geometric figure. (288–290)



$$\begin{aligned} V &= lwh \\ &= 7 \times 5 \times 3 \\ &= 105 \end{aligned}$$

Volume is 105 cu. in.



$$\begin{aligned} V &= \pi r^2 h \\ &= 3.14 \times 2^2 \times 7 \\ &= 87.92 \end{aligned}$$

Volume is 87.92 cu. in.

How to Change a Measurement to an Equivalent Measurement

Use a table of measures Use a table of measures to determine basic equivalent measurements. Then multiply the basic measures by the appropriate number. (249–251, 254)

$$5\frac{1}{2} \text{ yd.} = \underline{\hspace{1cm}} \text{ in.}$$

$$1 \text{ yd.} = \underline{\hspace{1cm}} 36 \text{ in.}$$

$$(5\frac{1}{2} \times 1) \text{ yd.} = (5\frac{1}{2} \times 36) \text{ in.}$$

$$5\frac{1}{2} \text{ yd.} = 198 \text{ in.}$$

$$321 \text{ cm.} = \underline{\hspace{1cm}} \text{ m.}$$

$$1 \text{ cm.} = .01 \text{ m.}$$

$$(321 \times 1) \text{ cm.} = (321 \times .01) \text{ m.}$$

$$321 \text{ cm.} = 3.21 \text{ m.}$$

$$13 \text{ min.} = \underline{\hspace{1cm}} \text{ sec.}$$

$$1 \text{ min.} = 60 \text{ sec.}$$

$$(13 \times 1) \text{ min.} = (13 \times 60) \text{ sec.}$$

$$13 \text{ min.} = 780 \text{ sec.}$$

How to Add and Subtract Measures

1. Use precision If two or more measurements have different precision, the sum or difference of their measures is recorded with the same precision as that of the least precise measurement. (260)

$$\begin{array}{r} 48.7 \text{ in.} \\ + 26.8 \text{ in.} \\ \hline 75.5 \text{ in.} \end{array}$$

$$\begin{array}{r} 36 \text{ yd.} \\ + 7.26 \text{ yd.} \\ \hline 43.26 \text{ yd.} \end{array}$$

$$\begin{array}{r} 21.38 \text{ ft.} \\ 8.5 \text{ ft.} \\ + 17.136 \text{ ft.} \\ \hline \end{array}$$

$$\begin{array}{r} 38.41 \text{ mi.} \\ - 22.426 \text{ mi.} \\ \hline \end{array}$$

$$\begin{array}{r} 48.7 \text{ in.} \\ - 26.8 \text{ in.} \\ \hline 21.9 \text{ in.} \end{array}$$

$$\begin{array}{r} 36 \text{ yd.} \\ - 7.26 \text{ yd.} \\ \hline 28.74 \text{ yd.} \end{array}$$

$$\begin{array}{r} 21.4 \text{ ft.} \\ 8.5 \text{ ft.} \\ + 17.1 \text{ ft.} \\ \hline 47.0 \text{ ft.} \end{array}$$

$$\begin{array}{r} 38.41 \text{ mi.} \\ - 22.43 \text{ mi.} \\ \hline 15.98 \text{ mi.} \end{array}$$

2. With more than one unit of measurement The precision of a measurement like 2 yd. 1 ft. 7 in. is indicated by the smallest unit used. (262)

$$\begin{array}{r} 11 \text{ hr.} \quad 38 \text{ min.} \quad 47 \text{ sec.} \\ + (5 \text{ hr.} \quad 29 \text{ min.} \quad 18 \text{ sec.}) \\ \hline 17 \text{ hr.} \quad 68 \text{ min.} \quad 65 \text{ sec.} \\ \quad (60 + 8) \text{ min.} \quad (60 + 5) \text{ sec.} \\ 17 \text{ hr.} \quad 8 \text{ min.} \quad 5 \text{ sec.} \end{array}$$

$$\begin{array}{r} 2 \text{ wk.} \quad 2 \text{ da.} \quad 7 \text{ hr.} \\ - (1 \text{ wk.} \quad 3 \text{ da.} \quad 12 \text{ hr.}) \\ \hline 1 \text{ wk.} \quad 5 \text{ da.} \quad 19 \text{ hr.} \end{array}$$

Multiplication

Terms and Concepts

1. Commutative property For any two numbers a and b , $ab = ba$. (58, 102, 114)

Whole numbers: $17 \times 85 = 85 \times 17$

Integers: $-5 \times +8 = +8 \times -5$

Rational numbers: $\frac{4}{5} \times \frac{3}{8} = \frac{3}{8} \times \frac{4}{5}$

2. Associative property For any numbers a , b , and c , $(a \times b) \times c = a \times (b \times c)$. (59, 102, 114)

Whole numbers: $(8 \times 5) \times 7 = 8 \times (5 \times 7)$

Integers: $(-3 \times +4) \times -2 = -3 \times (+4 \times -2)$

Rational numbers: $(\frac{1}{2} \times \frac{2}{3}) \times \frac{4}{5} = \frac{1}{2} \times (\frac{2}{3} \times \frac{4}{5})$

3. Identity number One is the identity number of multiplication. For every number n , $n \times 1 = n = 1 \times n$. (60, 115)

$32 \times 1 = 32$ $1 \times -7 = -7$ $\frac{5}{7} \times 1 = \frac{5}{7}$

4. Zero in multiplication If one of two factors is zero, then their product is zero. If a product is zero, then at least one of the factors is zero. For every number n , $n \times 0 = 0$ and if $8n = 0$, then $n = 0$. (60)

$7 \times 0 = 0$ $0 \times -18 = 0$

5. Inverse operation Division is the inverse operation of multiplication. If $a \times b = c$ and $b \neq 0$, then $a = \frac{c}{b}$. (66, 104)

$8 \times 9 = 72$ so $72 \div 9 = 8$

$-3 \times -4 = +12$ so $+12 \div -4 = -3$

6. Multiples of a number The product of any whole number n and another whole number is called a multiple of n . (69)

$3 \times 6 = 18$

18 is a multiple of 6.

18 is a multiple of 3.

How to Multiply Whole Numbers

1. Use the distributive property Rename either of the factors as a sum and then use the distributive property of multiplication over addition. (62)

$$\begin{aligned} 23 \times 14 &= 23 \times (10 + 4) \\ &= (23 \times 10) + (23 \times 4) \\ &= 230 + 92 \\ &= 322 \end{aligned}$$

$$\begin{aligned} 43 \times 21 &= (40 + 3) \times 21 \\ &= (40 \times 21) + (3 \times 21) \\ &= 840 + 63 \\ &= 903 \end{aligned}$$

2. Use the algorithm The multiplication algorithm is a convenient arrangement of numerals for finding the simplest numeral for a product. Observe how the distributive property is used in the algorithm. (64)

$$\begin{array}{r} 21 \\ \times 43 \\ \hline 63 \quad \text{---} \quad 3 \times 21 \\ 84 \quad \text{---} \quad 40 \times 21 \\ \hline 903 \quad \text{---} \quad (40 + 3) \times 21 \end{array}$$

How to Multiply Integers

1. Two positive integers The product of two positive integers is a positive integer. (102)

$$+7 \times +8 = +56$$

2. Two negative integers The product of two negative integers is a positive integer. (102)

$$-9 \times -6 = +54$$

3. A positive integer and a negative integer The product of two integers, one positive and the other negative, is a negative integer. (103)

$$-8 \times +3 = -24 \qquad +5 \times -6 = -30$$

How to Multiply Rational Numbers

1. In fraction form To find the product of two rational numbers named by fractions, find the product of the numerators and the product of the denominators. For rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. (112)

$$\frac{3}{4} \times \frac{5}{7} = \frac{3 \times 5}{4 \times 7} = \frac{15}{28}$$

2. With mixed numerals When one or more of the numbers is named by a mixed numeral, the product may be found by either of these procedures.

a. Use the distributive property (124)

$$\begin{aligned} 6 \times 4\frac{1}{3} &= 6 \times (4 + \frac{1}{3}) \\ &= (6 \times 4) + (6 \times \frac{1}{3}) \\ &= 24 + 2 \text{ or } 26 \end{aligned}$$

b. Change the mixed numerals to fractions (124)

$$\begin{aligned} 3\frac{2}{3} \times 2\frac{1}{2} &= \frac{11}{3} \times \frac{5}{2} \\ &= \frac{11 \times 5}{3 \times 2} \\ &= \frac{55}{6} \text{ or } 9\frac{1}{6} \end{aligned}$$

3. With decimals The following examples show how to multiply rational numbers named by decimals.

a. Placement of the decimal point

The number of digits to the right of the decimal point in the product numeral is the sum of the numbers of digits to the right of the decimal points in both factor numerals. (168)

$$\begin{array}{rcl} 1.47 & \longrightarrow & 2 \text{ digits} \\ \times .8 & \longrightarrow & 1 \text{ digit} \\ \hline 1.176 & \longleftarrow & 3 \text{ digits} \end{array}$$

b. Powers of ten When multiplying by a power of ten, the product numeral has the same digits as the numeral naming the other factor, but the placement of the decimal point changes. (170)

$$3.285 \times 100 = 328.5$$

c. Final zeros in a product If a product numeral has final 0's to the right of the decimal point, they may be omitted when naming the product. (171)

$$\begin{array}{r} 3.45 \\ \times .08 \\ \hline .2760 \text{ or } .276 \end{array}$$

Number and Numeration

Terms and Concepts

1. Non-positional numeration In a non-positional numeration system, the order in which the symbols are written is not important. (5,6,8)

Egyptian: $\Pi\Pi\Pi\Pi$ means $10+10+10+2$
 $\Pi\Pi\Pi\Pi$ means $2+10+10+10$

Roman: XVI means $10+5+1$
LIX means $50+(10-1)$

2. Positional numeration In a positional numeration system, the order of the symbols is used to denote the place value of each digit. (12-25)

Base ten:

138 means $(1 \times 10^2) + (3 \times 10^1) + (8 \times 1)$

Base five:

132_{five} means $(1 \times 5^2) + (3 \times 5^1) + (2 \times 1)$

Base seven:

324_{seven} means $(3 \times 7^2) + (2 \times 7^1) + (4 \times 1)$

Base two:

101_{two} means $(1 \times 2^2) + (0 \times 2^1) + (1 \times 1)$

3. Whole numbers A whole number is the number of a set. (51)

$\{0, 1, 2, 3, 4, 5, \dots\}$

4. Natural numbers The set of natural numbers is a special subset of the set of whole numbers. (51)

$\{1, 2, 3, 4, 5, \dots\}$

5. Prime number Any whole number greater than 1 that has only itself and 1 as factors is called a prime number. (81)

2, 3, 5, 7, 11, 13, 17, 19, 23, ...

6. Composite number Any whole number greater than 1 that is not a prime number is called a composite number. (81)

4, 6, 8, 9, 10, 12, 14, 15, 16, 18, ...

7. Integers The set of integers is defined as follows. (92)

$\{\dots, -3, -2, -1, 0, +1, +2, +3, \dots\}$

8. Rational numbers If a and b represent integers and $b \neq 0$, then $\frac{a}{b}$ represents a rational number. (110)

$-1, -\frac{2}{3}, -\frac{1}{4}, 0, +1, +\frac{5}{2}, +3\frac{3}{4}$

9. Extending positional numeration Positional numeration systems are easily extended to name any rational number. (155-157)

$\frac{3}{4} = .75$ $8\frac{1}{2} = 8.5$

10. Decimals for rational numbers Every rational number can be named by either a terminating or a repeating decimal. (162, 180)

Terminating: $\frac{4}{5} = .8$ $\frac{3}{8} = .375$

Repeating: $\frac{5}{3} = .\overline{6}$ $\frac{5}{11} = .\overline{45}$

Organizing Data

Terms and Concepts

1. Mode The mode of a set of data is the score or the scores that occur more frequently than any other score in the set. (295)

{5, 9, 12, 12, 15, 17, 23}

The mode of this set of data is 12.

2. Median The median of a set of data is the score that occupies the middle position when the scores are arranged numerically. (296)

{5, 6, 8, 9, 10, 15, 21}

The median of this set of data is 9.

3. Mean The arithmetic mean or the mean of a set of data is the average of the scores. (297)

{3, 4, 6, 6, 7, 10}

$$\frac{3+4+6+6+7+10}{6} = \frac{36}{6} \text{ or } 6$$

The mean of this set of data is 6.

4. Probability Let n represent the number of possible outcomes of an event. Let s represent the number of successful outcomes. Then the probability of a successful outcome is $\frac{s}{n}$. (302)

Suppose there are 7 possible outcomes of an event and 3 of these outcomes are successful outcomes. Then the probability of a successful outcome is $\frac{3}{7}$.

How to Find a Probability

1. Single outcome Use the definition of probability. (302)



Probability of drawing a blue marble in one draw is $\frac{3}{5}$.

2. Two different outcomes The probability of either of two outcomes is the sum of their individual probabilities. (304)



Probability of drawing a 4: $\frac{3}{7}$

Probability of drawing a 5: $\frac{2}{7}$

Probability of drawing a 4 or a 5:

$$\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$$

3. Successive outcomes The probability of two or more outcomes in succession is the product of the individual probabilities. (306-309)



Assume that once a marble is drawn it is not replaced.

Probability of blue first: $\frac{2}{4}$

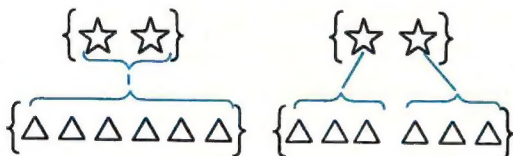
Probability of black second: $\frac{2}{3}$

Probability of blue first, black second: $\frac{2}{4} \times \frac{2}{3} = \frac{6}{20}$ or $\frac{3}{10}$

Ratio, Proportion, Per Cent

Terms and Concepts

1. Ratio A ratio is a correspondence between the numbers of two sets. (191)



Ratio of the number of stars to the number of triangles is $\frac{2}{6}$ or $\frac{1}{3}$.

Ratio of the number of triangles to the number of stars is $\frac{6}{2}$ or $\frac{3}{1}$.

2. Proportion A proportion expresses the equality of two ratios. (193)

$$\frac{2}{6} = \frac{1}{3}$$

$$\frac{x}{5} = \frac{14}{35}$$

3. Per cent If x stands for a number, then $x\%$ expresses the ratio of x to 100. (196)

$$13\% = \frac{13}{100} \text{ or } .13$$

$$342\% = \frac{342}{100} \text{ or } 3.42$$

How to Solve a Proportion

Use equivalent fractions Think of the ratios in the proportion as equivalent fractions. (193)

$$\frac{x}{5} = \frac{14}{35}$$

$$35x = 5 \times 14$$

$$x = \frac{5 \times 14}{35}$$

$$x = 2$$

How to Determine Equivalent Fractions, Decimals, and Per Cent

1. Per cent to decimal Change the per cent to a fraction with denominator of 100. Then simplify the fraction and use its decimal equivalent. (197)

$$3.17\% = \frac{317}{100} = \frac{300}{100} + \frac{17}{100} = 3.17$$

2. Fractional per cent to a fraction Multiply the fractional number named in the per cent notation by $\frac{1}{100}$. (198)

$$\frac{2}{3}\% = \frac{2}{3} \times \frac{1}{100} = \frac{2}{300} \text{ or } \frac{1}{150}$$

How to Use Per Cent

1. Per cent of a number Change the per cent to a fraction or a decimal and multiply. (200)

$$\begin{aligned} n &= 5\% \times 80 \\ &= .05 \times 80 \\ &= 4 \end{aligned}$$

$$\begin{aligned} n &= 5\% \times 80 \\ &= \frac{5}{100} \times 80 \\ &= 4 \end{aligned}$$

2. Use a proportion Every per cent problem can be solved by using a proportion. (202)

12 is what % of 48?

$$\frac{12}{48} = \frac{x}{100}$$

$$12 \times 100 = 48x$$

$$\frac{12 \times 100}{48} = x$$

$$\frac{100}{4} = x$$

$$25 = x$$

12 is 25% of 48.

Subtraction

Terms and Concepts

1. Inverse of addition For all numbers a , b , and c , if $a+b=c$, then $c-b=a$. (54)

$$8+34=42 \text{ so } 42-34=8$$

2. Subtracting an integer Subtracting an integer is equivalent to adding its opposite. (100)

$$+3-4=+3+4 \quad -9-+6=-9+6$$

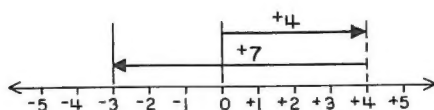
How to Subtract Whole Numbers

Rename the minuend Rename the minuend, if necessary, so that you can subtract in every place-value position. (56)

$$\begin{array}{r} 524 \\ -376 \\ \hline \end{array} \quad \begin{array}{r} \overset{4}{\cancel{5}} \overset{11}{2} \overset{14}{4} \\ -376 \\ \hline 148 \end{array}$$

How to Subtract Integers

1. On a number line Starting at the 0 mark, make the move indicated by the minuend. Then make the move indicated by the subtrahend, but in the *opposite direction* of that indicated by the subtrahend. (98)



$$+4-+7=-3$$

2. Change to addition To subtract an integer, add its opposite. (100)

$$-12-+7=-12+-7=-19$$

How to Subtract Rational Numbers

1. In fraction form The numbers may be named by fractions or by mixed numerals.

a. Same denominator If $\frac{a}{b}$ and $\frac{c}{b}$ represent rational numbers, then $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$. (146)

$$\frac{7}{12}-\frac{5}{12}=\frac{7-5}{12}=\frac{2}{12} \text{ or } \frac{1}{6}$$

b. Different denominators Rename the numbers so they have the same denominator. (146)

$$\begin{array}{r} \frac{5}{8}-\frac{1}{6}=\frac{15}{24}-\frac{4}{24} \\ =\frac{15-4}{24} \\ =\frac{11}{24} \end{array} \quad \begin{array}{r} \frac{5}{8} \\ -\frac{1}{6} \\ \hline \frac{11}{24} \end{array}$$

c. Mixed numerals Subtract the fractional parts and subtract the whole numbers. (148)

$$\begin{array}{r} 38\frac{2}{5} \\ -14\frac{1}{4} \\ \hline \end{array} \quad \begin{array}{r} 38\frac{8}{20} \\ -14\frac{5}{20} \\ \hline 24\frac{3}{20} \end{array}$$

2. With decimals Align the decimal points and subtract. (166)

$$\begin{array}{r} 380.562 \\ -218.927 \\ \hline 161.635 \end{array}$$